Habilitation à Diriger des Recherches

Réduction de sous-complexes de torsion[1] — *Torsion Subcomplex Reduction*[1]

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Abstract. /*Résumé.* Cette thèse décrit des travaux qui incorporent une technique appelée la réduction des sous-complexes de torsion (RST), et qui a été développée par l'auteur pour calculer la torsion dans la cohomologie de groupes discrets agissant sur des complexes cellulaires convenables. La RST permet de s'épargner des calculs sur la machine sur les complexes cellulaires, et d'accéder directement aux sous-complexes de torsion réduits, ce qui produit des résultats sur la cohomologie de groupes de matrices en termes de formules. La RST a déjà donné des formules générales pour la cohomologie des groupes de Coxeter tétraédraux, et, pour torsion impaire, de groupes SL_2 sur des entiers dans des corps de nombres arbitraires (en collaboration avec M. Wendt). Ces dernières formules ont permis à Wendt et l'auteur de raffiner la conjecture de Quillen.

D'ailleurs, des progrès ont été faits pour adapter la RST aux calculs de l'homologie de Bredon. En particulier pour les groupes de Bianchi, donnant toute leur K-homologie équivariante et, par le morphisme d'assemblage de Baum–Connes, la K-théorie de leur C^*-algèbres reduites, qui serait très dure à calculer directement.

En tant qu'une application collatérale, la RST a permis à l'auteur de fournir des formules de dimension pour la cohomologie orbi-espace de Chen–Ruan pour les orbi-espaces de Bianchi complexifiés, et de démontrer (en collaboration avec F. Perroni) la conjecture de Ruan sur la résolution crépante pour tous les orbi-espaces de Bianchi complexifiés.

Abstract. This thesis describes works involving a technique called Torsion Subcomplex Reduction (TSR), which was developed by the author for computing torsion in the cohomology of discrete groups acting on suitable cell complexes. TSR enables one to skip machine computations on cell complexes, and to access directly the reduced torsion subcomplexes, which yields results on the cohomology of matrix groups in terms of formulas. TSR has already yielded general formulas for the cohomology of the tetrahedral Coxeter groups as well as, at odd torsion, of SL_2 groups over arbitrary number rings (in joint work of M. Wendt and the author). The latter formulas have allowed Wendt and the author to refine the Quillen conjecture.

Furthermore, progress has been made to adapt TSR to Bredon homology computations. In particular for the Bianchi groups, yielding their equivariant K-homology, and, by the Baum–Connes assembly map, the K-theory of their reduced C^*-algebras, which would be very hard to compute directly.

As a side application, TSR has allowed the author to provide dimension formulas for the Chen–Ruan orbifold cohomology of the complexified Bianchi orbifolds, and to prove (jointly with F. Perroni) Ruan's crepant resolution conjecture for all complexified Bianchi orbifolds.

Table des matières

1. Introduction en français

Mon projet de recherche envisage de faire des progrès systématiques dans le calcul de certains invariants de groupes discrets. Le progrès que j'ai déjà fait s'appuie sur la technique de la *réduction des sous-complexes de torsion* pour l'étude de groupes discrets, que j'ai d'abord mise en oeuvre dans [**46**] pour une classe spécifique de groupes discrets : les groupes de Bianchi, pour lesquels la méthode a fourni toute l'homologie au dessus de la dimension cohomologique virtuelle. Des éléments de cette technique avaient déjà été utilisés avant par Soulé pour un groupe modulaire [**64**] ; et des versions ad hoc de la méthode avaient été mis en oeuvre par Mislin et puis par Henn [**23**]. Ayant réussi à mettre la technique dans un cadre assez général [**45**], j'ai pour projet de l'appliquer à un ensemble de classes de groupes aussi large que possible.
Il convient de donner quelques exemples où la méthode a déjà donné de bons résultats :

- Les groupes de Bianchi,
- Les groupes de Coxeter,
- Les groupes SL_2 sur des anneaux de nombres arbitraires.

Les groupes de Bianchi. Dans le cas des groupes de Bianchi (groupes PSL_2 sur les anneaux quadratiques imaginaires), la technique de réduction des sous-complexes de torsion m'a permis de trouver une description de l'anneau de cohomologie de ces groupes en termes de quantités élémentaires de la théorie des nombres [**45**]. L'étape décisive a été d'extraire, à l'aide de la réduction des sous-complexes de torsion, les informations essentielles des modèles géométriques, et puis de les détacher complètement du modèle. J'ai donc pu démontrer que toutes ces informations sont contenues dans les *graphes des classes de conjugaison*, que je construis à cette fin pour un groupe arbitraire en partant de son système de sous-groupes finis modulo l'opération de conjugaison. Un des aspects que je me propose ainsi d'étudier dans ce projet concerne le comportement des graphes des classes de conjugaison pour les autres classes de groupes arithmétiques étudiés.

Les groupes de Coxeter. Rappelons que les groupes de Coxeter sont engendrés par des réflections ; et leur homologie consiste uniquement en de la torsion. La technique de réduction des sous-complexes de torsion permet ainsi d'emblée d'obtenir toute la torsion homologique de tous les groupes de Coxeter tétraédraux pour tous les nombres premiers impairs, dans une formule générale et aussi en termes de tableaux explicites [**45**].

Les groupes SL_2 sur des anneaux de nombres arbitraires. En collaboration avec Matthias Wendt, j'ai établi des formules pour la cohomologie de Farrell-Tate à coefficients de torsion impaire de tous les groupes $\mathrm{SL}_2(A)$, où A est un anneau de S-entiers dans un corps de nombres arbitraire [**54**]. Wendt a aussi étendu ceci aux cas où A est un anneau de fonctions sur une courbe affine lisse sur un corps algébriquement clos. Ces deux résultats ensemble ont permis à Wendt de trouver une version raffinée de la conjecture de Quillen, qui tient compte de tous les types de contre-exemples connus [**55**]. Donc s'il n'existe pas de contre-exemple de type complètement nouveau à la conjecture de Quillen, la conjecture de Quillen–Wendt doit être vraie.

Prochaine étape : Les groupes modulaires. Les groupes modulaires $\mathrm{SL}_n(\mathbb{Z})$ sont assez proches des groupes de Bianchi et présentent un grand intérêt, car ils apparaissent dans de nombreuses disciplines mathématiques. Considérant de futurs développements de la technique de réduction des sous-complexes de torsion, il semble donc important de les traiter. Par contre, il y a un manque de modèles calculatoires préservant la torsion pour cette classe de groupes : on ne dispose de tels modèles que pour $\mathrm{SL}_2(\mathbb{Z})$, où la torsion admet une structure très simple, et pour $\mathrm{SL}_3(\mathbb{Z})$, où un modèle cellulaire célèbre a été élaboré par Soulé [**64**]. Ce dernier modèle admet uniquement des stabilisateurs de cellules qui fixent ces dernières point par point. Cette propriété n'a pu être atteinte ni par le modèle de Ash [**3**–**7**], ni par le modèle des polytopes de Voronoï [**21**] pour $\mathrm{SL}_n(\mathbb{Z})$. Le modèle de Soulé a été étudié et généralisé par Hans-Werner Henn [**25**], mais n'a été mis en pratique que jusqu'à $\mathrm{SL}_3(\mathbb{Z}[\frac{1}{2}])$. Récemment, ce problème a été résolu par un algorithme développé par Tuan Anh Bui et moi-même, qui permet de transformer les complexes cellulaires donnés d'une manière efficace en des complexes cellulaires avec la propriété désirée. Ensuite, ma technique de réduction des sous-complexes de torsion s'applique.

Application à la conjecture de Baum/Connes. En se servant de complexes cellulaires avec une action des stabilisateurs sans inversions de cellules, on peut calculer l'homologie de Bredon des groupes arithmétiques en question, pour en déduire leur K-homologie équivariante. Ceci a été fait par Sanchez-Garcia pour $\mathrm{SL}_3(\mathbb{Z})$ [**58**] et des groupes de Coxeter [**59**], et je l'ai effectué pour des groupes de Bianchi [**42**]. La K-homologie équivariante est le côté géométrique-topologique de la conjecture de Baum/Connes : Baum et Connes construisent un homomorphisme de la K-homologie équivariante à la K-théorie des C^*-algèbres réduites d'un groupe donné. Leur conjecture dit que cet homomorphisme, appelé le morphisme d'assemblage, est un isomorphisme. La conjecture de Baum/Connes implique plusieurs conjectures importantes en topologie, en géométrie, en algèbre et en analyse fonctionnelle : quand le morphisme d'assemblage est surjectif, le groupe vérifie la conjecture de Kaplansky/Kadison sur les idempotents ; quand le morphisme d'assemblage est injectif, le groupe vérifie la conjecture forte de Novikov et une partie de la conjecture de Gromov/Lawson/Rosenberg. Il est donc intéressant d'obtenir la K-homologie équivariante des groupes $\mathrm{SL}_n(\mathbb{Z})$, $n \geq 4$, pour lequels la conjecture de Baum/Connes est ouverte.

2. Achievements concerning torsion subcomplex reduction

2.1. Organisation of the thesis. This thesis is a survey paper on a selection of works of the author. Therefore, it contains no new result itself, but attempts to quote correctly the relevant original results. The focus of this selection are the techniques presented in Section 3.
The motivation for those techniques are the results described in Sections 2.3 and 2.4, as well as the computations for algebraic K-theory via Farrell cohomology begun by Schwermer and Vogtmann [**61**]. Future work involving planned improvements of the techniques is sketched in Section 5. Published work of the author not involving the techniques is only summarized Section 4.

2.2. Background. Our objects of study are discrete groups Γ such that Γ admits a torsion-free subgroup of finite index. By a theorem of Serre [**62**], all the torsion-free subgroups of finite index in Γ have the same cohomological dimension; this dimension is called the virtual cohomological dimension (abbreviated vcd) of Γ. Above the vcd, the (co)homology of a discrete group is determined by its system of finite subgroups. We are going to discuss it in terms of Farrell–Tate cohomology (which we will by now just call Farrell cohomology). The Farrell cohomology $\widehat{\mathrm{H}}^q$ is identical to group cohomology H^q in all degrees q above the vcd, and extends in lower degrees to a cohomology theory of the system of finite subgroups. Details are elaborated in [**11**, chapter X]. So for instance considering the Coxeter groups, the virtual cohomological dimension of all of which vanishes, their Farrell cohomology is identical to all of their group cohomology. In Section 3.1, we will introduce a method of how to explicitly determine the Farrell cohomology : By reducing torsion sub-complexes.

2.3. Statement of the results. Let me start with results related to the novel technique of *torsion subcomplex reduction*, which I have developed. It is a technique for the study of discrete groups Γ, giving easier access to the cohomology of the latter at a fixed prime ℓ and above the virtual cohomological dimension, by extracting the relevant portion of the equivariant spectral sequence and then simplifying it. Instead of having to work with a full cellular complex X with a nice Γ-action, the technique inputs only an often lower-dimensional subcomplex of X, and reduces it to a small number of cells.

I first used torsion subcomplex reduction in [**46**] for a specific class of arithmetic groups, the Bianchi groups, for which my method yielded all of the homology above the virtual cohomological dimension. Some elements of this technique had already been used by Soulé for a modular group [**64**]; and were used by Mislin and Henn as a set of ad hoc tricks. After rediscovering these ad hoc tricks, I had success in putting them into a general framework [**45**]. The advantage of using a systematic technique rather than a set of ad-hoc tricks is that instead of merely allowing for isolated ad-hoc example calculations, it becomes possible to find general formulas, as I did for instance for the entire family of the Bianchi groups.

It is convenient to give some examples of where the technique of torsion subcomplex reduction has already produced good results:

- The Bianchi groups,
- The Coxeter groups,
- The SL_2 groups over arbitrary number rings.

In this section, I would like to outline the results. Then in Section 3, I will provide a more detailed look at these methods.

2.3.1. *The Bianchi groups.* In the case of the PSL_2 groups over rings of imaginary quadratic integers (known as the Bianchi groups), the torsion subcomplex reduction technique has permitted me to find a description of the cohomology ring of these groups in terms of elementary number-theoretic quantities [**45**]. The key step has been to extract, using torsion subcomplex reduction, the essential information about the geometric models, and then to detach the cohomological information completely from the model. I was hence able to show that this information is contained in objects which I call "conjugacy classes graphs", which I construct for an arbitrary group from its system of conjugacy classes of finite subgroups.

2.3.2. *The Coxeter groups.* Recall that the Coxeter groups are generated by reflections, and their homology consists solely of torsion. Thus, torsion subcomplex reduction allows one to obtain all homology groups for all of the tetrahedral Coxeter groups at all odd prime numbers, in terms of a general formula [**45**].

2.4. State of the art on the future objectives. In the following, Section 2.4.m, with m running from 1 to 5, describes the state of the art on Objective 5.m defined below.

2.4.1. *The SL_2 groups over arbitrary number rings.* In joint work [**54**], Matthias Wendt and I have established a complete description of the Farrell–Tate cohomology with odd torsion coefficients for all groups $\mathrm{SL}_2(\mathcal{O}_{K,S})$, where $\mathcal{O}_{K,S}$ is the ring of S-integers in an arbitrary number field K at an arbitrary non-empty finite set S of places of K containing the infinite places, based on an explicit description of conjugacy classes of finite cyclic subgroups and their normalizers in $\mathrm{SL}_2(\mathcal{O}_{K,S})$.

Our statement uses the following notation. Let ℓ be an odd prime number different from the characteristic of K. In the situation where, for ζ_ℓ some primitive ℓ-th root of unity, $\zeta_\ell + \zeta_\ell^{-1} \in K$, we will abuse notation and write $\mathcal{O}_{K,S}[\zeta_\ell]$ to mean the ring $\mathcal{O}_{K,S}[T]/(T^2 - (\zeta_\ell + \zeta_\ell^{-1})T + 1)$. Moreover, we denote the norm maps for class groups and units by

$$\mathrm{Nm}_0 : \widetilde{\mathrm{K}_0}(\mathcal{O}_{K,S}[\zeta_\ell]) \to \widetilde{\mathrm{K}_0}(\mathcal{O}_{K,S}) \qquad \text{and} \qquad \mathrm{Nm}_1 : \mathcal{O}_{K,S}[\zeta_\ell]^\times \to \mathcal{O}_{K,S}^\times.$$

Denote by $M_{(\ell)}$ the ℓ-primary part of a module M; by $N_G(\Gamma)$ the normalizer of Γ in G; and by $\widehat{\mathrm{H}}^\bullet$ Farrell cohomology (cf. Section 2.2).

Theorem 1 ([**54**])**.**

(1) $\widehat{\mathrm{H}}^\bullet(\mathrm{SL}_2(\mathcal{O}_{K,S}), \mathbb{F}_\ell) \neq 0$ *if and only if* $\zeta_\ell + \zeta_\ell^{-1} \in K$ *and the Steinitz class* $\det_{\mathcal{O}_{K,S}}(\mathcal{O}_{K,S}[\zeta_\ell])$ *is contained in the image of the norm map* Nm_0.

(2) Assume the condition in (1) is satisfied. The set $\mathcal{C}_\ell$ *of conjugacy classes of order* ℓ *elements in* $\mathrm{SL}_2(\mathcal{O}_{K,S})$ *sits in an extension*

$$1 \to \operatorname{coker} \mathrm{Nm}_1 \to \mathcal{C}_\ell \to \ker \mathrm{Nm}_0 \to 0.$$

The set $\mathcal{K}_\ell$ *of conjugacy classes of order* ℓ *subgroups of* $\mathrm{SL}_2(\mathcal{O}_{K,S})$ *can be identified with the quotient* $\mathcal{K}_\ell = \mathcal{C}_\ell / \mathrm{Gal}(K(\zeta_\ell)/K)$. *There is a direct sum decomposition*

$$\widehat{\mathrm{H}}^\bullet(\mathrm{SL}_2(\mathcal{O}_{K,S}), \mathbb{F}_\ell) \cong \bigoplus_{[\Gamma] \in \mathcal{K}_\ell} \widehat{\mathrm{H}}^\bullet(N_{\mathrm{SL}_2(\mathcal{O}_{K,S})}(\Gamma), \mathbb{F}_\ell)$$

which is compatible with the ring structure, i.e., the Farrell-Tate cohomology ring of $\mathrm{SL}_2(\mathcal{O}_{K,S})$ *is a direct sum of the sub-rings for the normalizers* $N_{\mathrm{SL}_2(\mathcal{O}_{K,S})}(\Gamma)$.

(3) If the class of Γ *is not* $\mathrm{Gal}(K(\zeta_\ell)/K)$*-invariant, then*

$$N_{\mathrm{SL}_2(\mathcal{O}_{K,S})}(\Gamma) \cong \ker \mathrm{Nm}_1 .$$

There is a degree 2 *cohomology class* a_2 *and a ring isomorphism*

$$\widehat{\mathrm{H}}^\bullet(\ker \mathrm{Nm}_1, \mathbb{Z})_{(\ell)} \cong \mathbb{F}_\ell[a_2, a_2^{-1}] \otimes_{\mathbb{F}_\ell} \bigwedge (\ker \mathrm{Nm}_1) .$$

In particular, this is a free module over the subring $\mathbb{F}_\ell[a_2^2, a_2^{-2}]$.

(4) If the class of Γ *is* $\mathrm{Gal}(K(\zeta_\ell)/K)$*-invariant, then there is an extension*

$$0 \to \ker \mathrm{Nm}_1 \to N_{\mathrm{SL}_2(\mathcal{O}_{K,S})}(\Gamma) \to \mathbb{Z}/2 \to 1.$$

There is a ring isomorphism

$$\widehat{\mathrm{H}}^\bullet(N_{\mathrm{SL}_2(\mathcal{O}_{K,S})}(\Gamma), \mathbb{Z})_{(\ell)} \cong \left(\mathbb{F}_\ell[a_2, a_2^{-1}] \otimes_{\mathbb{F}_\ell} \bigwedge (\ker \mathrm{Nm}_1)\right)^{\mathbb{Z}/2},$$

with the $\mathbb{Z}/2$-action given by multiplication with -1 on a_2 and $\ker \mathrm{Nm}_1$*. In particular, this is a free module over the subring*

$$\mathbb{F}_\ell[a_2^2, a_2^{-2}] \cong \widehat{\mathrm{H}}^\bullet(D_{2\ell}, \mathbb{Z})_{(\ell)}.$$

(5) The restriction map induced from the inclusion $\mathrm{SL}_2(\mathcal{O}_{K,S}) \to \mathrm{SL}_2(\mathbb{C})$ *maps the second Chern class c_2 to the sum of the elements a_2^2 in all the components.*

Wendt has furthermore extended this investigation to the cases of SL_2 over the ring of functions on a smooth affine curve over an algebraically closed field [**68**].

2.4.2. *Investigation of the refined Quillen conjecture.* The Quillen conjecture on the cohomology of arithmetic groups has spurred a great deal of mathematics (see the pertinent monograph [**28**]). Using our Farrell–Tate cohomology computations, Matthias Wendt and I have established further positive cases for the Quillen conjecture for SL_2. In detail, the original conjecture of 1971 [**40**] is as follows for GL_n.

Conjecture 2 (Quillen)**.** *Let ℓ be a prime number. Let K be a number field with $\zeta_\ell \in K$, and S a finite set of places containing the infinite places and the places over ℓ. Then the natural inclusion $\mathcal{O}_{K,S} \hookrightarrow \mathbb{C}$ makes* $\mathrm{H}^\bullet(\mathrm{GL}_n(\mathcal{O}_{K,S}), \mathbb{F}_\ell)$ *a free module over the cohomology ring* $\mathrm{H}^\bullet_{\mathrm{cts}}(\mathrm{GL}_n(\mathbb{C}), \mathbb{F}_\ell)$.

While there are counterexamples to the original version of the conjecture, it holds true in many other cases. From the first counterexamples through the present, the conjecture has kept researchers interested in determining its range of validity [**2**].

Positive cases in which the conjecture has been established are $n = \ell = 2$ by Mitchell [**37**], $n = 3$, $\ell = 2$ by Henn [**25**], and $n = 2$, $\ell = 3$ by Anton [**1**].

On the other hand, cases where the Quillen conjecture is known to be false can all be traced to [**26**, remark on p. 51], which shows that Quillen's conjecture for $\mathrm{GL}_n(\mathbb{Z}[1/2])$ implies that the restriction map

$$\mathrm{H}^\bullet(\mathrm{GL}_n(\mathbb{Z}[1/2]), \mathbb{F}_2) \to \mathrm{H}^\bullet(\mathrm{T}_n(\mathbb{Z}[1/2]), \mathbb{F}_2)$$

from $\mathrm{GL}_n(\mathbb{Z}[1/2])$ to the subgroup $\mathrm{T}_n(\mathbb{Z}[1/2])$ of diagonal matrices is injective. Non-injectivity of the restriction map has been shown by Dwyer [**20**] for $n \geq 32$ and $\ell = 2$. Dwyer's bound was subsequently improved by Henn and Lannes to $n \geq 14$. At the prime $\ell = 3$, Anton proved non-injectivity for $n \geq 27$, cf. [**1**].

Matthias Wendt's and my contribution is that we can determine precisely the module structure above the virtual cohomological dimension; this has allowed us to relate the Quillen conjecture for SL_2 to statements about Steinberg homology.

This, together with the results of [**68**], has allowed us to find a refined version of the Quillen conjecture, which keeps track of all the types of known counter-examples to the original Quillen conjecture:

Conjecture 3 (Refined Quillen conjecture [**55**])**.** *Let K be a number field. Fix a prime ℓ such that $\zeta_\ell \in K$, and an integer $n < \ell$. Assume that S is a set of places containing the infinite places and the places lying over ℓ. If each cohomology class of $\mathrm{GL}_n(\mathcal{O}_{K,S})$ is detected on some finite subgroup, then $\mathrm{H}^\bullet(\mathrm{GL}_n(\mathcal{O}_{K,S}), \mathbb{F}_\ell)$ is a free module over the image of the restriction map*

$$\mathrm{H}^\bullet_{\mathrm{cts}}(\mathrm{GL}_n(\mathbb{C}), \mathbb{F}_\ell) \to \mathrm{H}^\bullet(\mathrm{GL}_n(\mathcal{O}_{K,S}), \mathbb{F}_\ell).$$

For SL_2, we have made the following use of our description of the Farrell–Tate cohomology of SL_2 over rings of S-integers.

Corollary 4 (Corollary to Theorem 1)**.** *Let K be a number field, let S be a finite set of places containing the infinite ones, and let ℓ be an odd prime.*

(1) The original Quillen conjecture holds for group cohomology $\mathrm{H}^\bullet(\mathrm{SL}_2(\mathcal{O}_{K,S}), \mathbb{F}_\ell)$ above the virtual cohomological dimension.
(2) The refined Quillen conjecture holds for Farrell–Tate cohomology $\widehat{\mathrm{H}}^\bullet(\mathrm{SL}_2(\mathcal{O}_{K,S}), \mathbb{F}_\ell)$.

2.4.3. *Adaptation of the technique to groups with non-trivial centre.* Ethan Berkove and I have extended my technique of torsion subcomplex reduction, which originally was designed for groups with trivial centre (e.g., PSL_2), to groups with non-trivial centre (e.g., SL_2). This way, in [**8**], we have determined the 2-torsion in the cohomology of the SL_2 groups over imaginary quadratic number rings $\mathcal{O}_{-m}$ in $\mathbb{Q}(\sqrt{-m})$, based on their action on hyperbolic 3-space $\mathcal{H}^3$.

For instance, we get the following result in the case where the quotient of the 2–torsion subcomplex has the shape $\bullet\!\!-\!\!\bullet$, which is equivalent to the following three conditions (cf. [**45**]): $m \equiv 3 \bmod 8$, the field $\mathbb{Q}(\sqrt{-m})$ has precisely one finite ramification place over $\mathbb{Q}$, and the ideal class number of the totally real number field $\mathbb{Q}(\sqrt{m})$ is 1. Under these assumptions, our cohomology ring has the following dimensions:

$$\dim_{\mathbb{F}_2} \mathrm{H}^q(\mathrm{SL}_2(\mathcal{O}_{-m}); \mathbb{F}_2) = \begin{cases} \beta^1 + \beta^2, & q = 4k+5, \\ \beta^1 + \beta^2 + 2, & q = 4k+4, \\ \beta^1 + \beta^2 + 3, & q = 4k+3, \\ \beta^1 + \beta^2 + 1, & q = 4k+2, \\ \beta^1, & q = 1, \end{cases}$$

where $\beta^q := \dim_{\mathbb{F}_2} \mathrm{H}^q({}_{\mathrm{SL}_2(\mathcal{O}_{-m})}\backslash\mathcal{H}^3;\ \mathbb{F}_2)$. Let $\beta_1 := \dim_{\mathbb{Q}} \mathrm{H}_1({}_{\mathrm{SL}_2(\mathcal{O}_{-m})}\backslash\mathcal{H}^3;\ \mathbb{Q})$. For all absolute values of the discriminant less than 296, numerical calculations yield $\beta^2 + 1 = \beta^1 = \beta_1$. In this range, the numbers m subject to the above dimension formula and β_1 are given as follows (the Betti numbers are taken from [**49**]).

m	11	19	43	59	67	83	107	131	139	163	179	211	227	251	283
β_1	1	1	2	4	3	5	6	8	7	7	10	10	12	14	13

This result is a consequence of Theorem 26, combined with Lemma 27 below.

2.4.4. *Application to equivariant K-homology.* In a recent paper [**47**], I have, for the Bianchi groups, adapted the torsion subcomplex reduction technique from group homology to Bredon homology with coefficients in the complex representation rings, and with respect to the family of finite subgroups. This has led me to the following formulas for this Bredon homology, and by the Atiyah–Hirzebruch spectral sequence, to the formulas below for equivariant K-homology of the Bianchi groups acting on their classifying space for proper actions.

Theorem 5. *Let Γ be a Bianchi group or any one of its subgroups. Then the Bredon homology $\mathrm{H}_n^{\mathfrak{Fin}}(\Gamma;\ R_{\mathbb{C}})$ splits as a direct sum over*

(1) the orbit space homology $\mathrm{H}_n(\underline{\mathrm{B}}\Gamma;\ \mathbb{Z})$,
(2) a submodule $\mathrm{H}_n(\Psi_\bullet^{(2)})$ determined by the reduced 2-torsion subcomplex of $(\underline{\mathrm{E}}\Gamma, \Gamma)$
(3) and a submodule $\mathrm{H}_n(\Psi_\bullet^{(3)})$ determined by the reduced 3-torsion subcomplex of $(\underline{\mathrm{E}}\Gamma, \Gamma)$.

These submodules are given as follows.

Except for the Gaussian and Eisenstein integers, which can easily be treated ad hoc [**42**], all the rings of integers of imaginary quadratic number fields admit as only units $\{\pm 1\}$. In the latter case, we call $\mathrm{PSL}_2(\mathcal{O}_{-m})$ a *Bianchi group with units* $\{\pm 1\}$.

Theorem 6. *The 2-torsion part of the Bredon complex of a Bianchi group Γ with units $\{\pm 1\}$ has homology*

$$\mathrm{H}_n(\Psi_\bullet^{(2)}) \cong \begin{cases} \mathbb{Z}^{z_2} \oplus (\mathbb{Z}/2)^{\frac{d_2}{2}}, & n = 0, \\ \mathbb{Z}^{o_2}, & n = 1, \\ 0, & \text{otherwise}, \end{cases}$$

where z_2 counts the number of conjugacy classes of subgroups of type $\mathbb{Z}/2$ in Γ, o_2 counts the conjugacy classes of type $\mathbb{Z}/2$ in Γ which are not contained in any 2-dihedral subgroup, and d_2 counts the number of 2-dihedral subgroups, whether or not they are contained in a tetrahedral subgroup of Γ.

Theorem 7. *The* 3*-torsion part of the Bredon complex of a Bianchi group* Γ *with units* $\{\pm 1\}$ *has homology*

$$\mathrm{H}_n(\Psi_\bullet^{(3)}) \cong \begin{cases} \mathbb{Z}^{2o_3+\iota_3}, & n = 0 \text{ or } 1, \\ 0, & \text{otherwise,} \end{cases}$$

where amongst the subgroups of type $\mathbb{Z}/3$ *in* Γ*,* o_3 *counts the number of conjugacy classes of those of them which are not contained in any* 3*-dihedral subgroup, and* ι_3 *counts the conjugacy classes of those of them which are contained in some* 3*-dihedral subgroup in* Γ*.*

There are formulas for o_2, z_2, d_2, o_3 and ι_3 in terms of elementary number-theoretic quantities [**29**], which are readily computable by machine [**45**, appendix]. See Table 2 for how they relate to the types of connected components of torsion subcomplexes.

We deduce the following formulas for the equivariant K-homology of the Bianchi groups. Note for this purpose that for a Bianchi group Γ, there is a model for $\underline{\mathrm{E}}\Gamma$ of dimension 2, so $\mathrm{H}_2(\underline{\mathrm{B}}\Gamma;\, \mathbb{Z}) \cong \mathbb{Z}^{\beta_2}$ is torsion-free. Note also that the naive Euler characteristic of the Bianchi groups vanishes (again excluding the two special cases of Gaussian and Eisensteinian integers), that is, for $\beta_i = \dim \mathrm{H}_i(\underline{\mathrm{B}}\Gamma;\, \mathbb{Q})$ we have $\beta_0 - \beta_1 + \beta_2 = 0$ and $\beta_0 = 1$.

Corollary 8. *For any Bianchi group* Γ *with units* $\{\pm 1\}$*, the short exact sequence linking Bredon homology and equivariant* K*-homology splits into*

$$K_0^\Gamma(\underline{\mathrm{E}}\Gamma) \cong \mathbb{Z} \oplus \mathbb{Z}^{\beta_2} \oplus \mathbb{Z}^{z_2} \oplus (\mathbb{Z}/2)^{\frac{d_2}{2}} \oplus \mathbb{Z}^{2o_3+\iota_3}.$$

Furthermore, $K_1^\Gamma(\underline{\mathrm{E}}\Gamma) \cong \mathrm{H}_1(\underline{\mathrm{B}}\Gamma;\, \mathbb{Z}) \oplus \mathbb{Z}^{o_2} \oplus \mathbb{Z}^{2o_3+\iota_3}$*.*

2.4.5. *Chen–Ruan orbifold cohomology of the complexified orbifolds.* Jointly with Fabio Perroni, I have studied orbifolds X given by the induced action of the Bianchi groups on a complexification of $\mathrm{SL}_2(\mathbb{C})/\mathrm{SU}_2$. For these orbifolds, I have computed the Chen–Ruan Orbifold Cohomology as follows.

Theorem 9 ([**38**])**.** *Let* Γ *be a finite index subgroup in a Bianchi group (except over the Gaussian or Eisensteinian integers). Denote by* λ_{2n} *the number of conjugacy classes of cyclic subgroups of order* n *in* Γ*. Denote by* λ_{2n}^* *the cardinality of the subset of conjugacy classes which are contained in a dihedral subgroup of order* $2n$ *in* Γ*. Then,*

$$\mathrm{H}^d_{orb}\left([(\mathrm{SL}_2(\mathbb{C})/\mathrm{SU}_2)_\mathbb{C}/\Gamma]\right) \cong \mathrm{H}^d\left((\mathrm{SL}_2(\mathbb{C})/\mathrm{SU}_2)/_\Gamma;\, \mathbb{Q}\right) \oplus \begin{cases} \mathbb{Q}^{\lambda_4+2\lambda_6-\lambda_6^*}, & d = 2, \\ \mathbb{Q}^{\lambda_4-\lambda_4^*+2\lambda_6-\lambda_6^*}, & d = 3, \\ 0, & \text{otherwise.} \end{cases}$$

The (co)homology of the quotient space $(\mathrm{SL}_2(\mathbb{C})/\mathrm{SU}_2)/_{\Gamma}$ has been computed numerically for a large range of Bianchi groups [**66**], [**60**], [**49**]; and bounds for its Betti numbers have been given in [**30**]. Krämer [**29**] has determined number-theoretic formulas for the numbers λ_{2n} and λ^*_{2n} of conjugacy classes of finite subgroups in the Bianchi groups.

Building on this, Perroni and I have established the following result (not yet published [**38**]).

Theorem 10.
Let $(\mathrm{SL}_2(\mathbb{C})/\mathrm{SU}_2)_{\mathbb{C}}/_{\Gamma}$ *be the coarse moduli space of* $[(\mathrm{SL}_2(\mathbb{C})/\mathrm{SU}_2)_{\mathbb{C}}/\Gamma]$*; and let* $Y \to (\mathrm{SL}_2(\mathbb{C})/\mathrm{SU}_2)_{\mathbb{C}}/_{\Gamma}$ *be a crepant resolution of* $(\mathrm{SL}_2(\mathbb{C})/\mathrm{SU}_2)_{\mathbb{C}}/_{\Gamma}$*.*
Then there is an isomorphism as graded $\mathbb{C}$*-algebras between the Chen-Ruan cohomology ring of* $[(\mathrm{SL}_2(\mathbb{C})/\mathrm{SU}_2)_{\mathbb{C}}/\Gamma]$ *and the singular cohomology ring of* Y*:*

$$(H^*_{\mathrm{CR}}([(\mathrm{SL}_2(\mathbb{C})/\mathrm{SU}_2)_{\mathbb{C}}/\Gamma]), \cup_{\mathrm{CR}}) \cong (H^*(Y), \cup) \ .$$

The Chen–Ruan orbifold cohomology is conjectured by Ruan to match the quantum corrected classical cohomology ring of a crepant resolution for the orbifold. We have proved furthermore that the Gromov-Witten invariants involved in the definition of the quantum corrected cohomology ring of $Y \to (\mathrm{SL}_2(\mathbb{C})/\mathrm{SU}_2)_{\mathbb{C}}/_{\Gamma}$ vanish. Hence, Perroni and I have deduced the following.

Corollary 11. *Ruan's crepant resolution conjecture holds true for the complexified Bianchi orbifolds* $[(\mathrm{SL}_2(\mathbb{C})/\mathrm{SU}_2)_{\mathbb{C}}/\Gamma]$*.*

2.5. Publications concerning torsion subcomplex reduction.

- Alexander D. Rahm, *The homological torsion of* PSL_2 *of the imaginary quadratic integers*, Transactions of the AMS, volume 365 (2013), pp. 1603–1635.

  We reveal a correspondence between the homological torsion of the Bianchi groups and new geometric invariants, which are effectively computable thanks to their action on hyperbolic space. We develop the basics of torsion subcomplex reduction in order to obtain these invariants. We use it to explicitly compute the integral group homology of the Bianchi groups. Furthermore, this correspondence facilitates the computation of the equivariant K-homology of the Bianchi groups. By the Baum–Connes conjecture, which is satisfied by the Bianchi groups, we obtain the K-theory of their reduced C*-algebras in terms of isomorphic images of their equivariant K-homology.

- Alexander D. Rahm, *Homology and K-theory of the Bianchi groups — Homologie et K-théorie des groupes de Bianchi*, Comptes Rendus Mathématique de l'Académie des Sciences - Paris, volume 349 (2011). pp. 615–619.

  Announcement note of the above paper. Provides a French version.

- Alexander D. Rahm, *Accessing the cohomology of discrete groups above their virtual cohomological dimension*, Journal of Algebra, Volume 404, 15 February 2014, pp. 152–175.

  We introduce a method to explicitly determine the Farrell–Tate cohomology of discrete groups. We apply this method to the Coxeter triangle and tetrahedral groups as well as to the Bianchi groups, i.e. $\mathrm{PSL}_2(\mathcal{O})$ for $\mathcal{O}$ the ring of integers in an imaginary quadratic number field, and to their finite index subgroups. We show that the Farrell–Tate cohomology of the Bianchi groups is completely determined by the numbers of conjugacy classes of finite subgroups. In fact, our access to Farrell–Tate cohomology allows us to detach the information about it from geometric models for the Bianchi groups and to express it only in terms of the group structure. Formulae for the numbers of conjugacy classes of finite subgroups have been determined in a thesis of Krämer, in terms of elementary number-theoretic information on $\mathcal{O}$. An evaluation of these formulas for a large number of Bianchi groups is provided numerically in the electronically released appendix to this paper. Our new insights about their homological torsion allow us to give a conceptual description of the cohomology ring structure of the Bianchi groups.

- Ethan Berkove and Alexander D. Rahm, *The mod 2 cohomology rings of SL_2 of the imaginary quadratic integers*. With an appendix by Aurel Page. Journal of Pure and Applied Algebra, Volume 220 (2016), no. 3, pp. 944–975.

  We establish general dimension formulas for the second page of the equivariant spectral sequence of the action of the SL_2 groups over imaginary quadratic integers on their associated symmetric space. On the way, we extend the torsion subcomplex reduction technique to cases where the kernel of the group action is non-trivial. Using the equivariant and Lyndon–Hochschild–Serre spectral sequences, we investigate the second page differentials and show how to obtain the mod 2 cohomology rings of our groups from this information.

- Alexander D. Rahm and Matthias Wendt, *A refinement of a conjecture of Quillen*, Comptes Rendus Mathématique de l'Académie des Sciences, Volume 353, Issue 9, September 2015, pp. 779–784.

  We present some new results on the cohomology of a large range of SL_2-groups in degrees above the virtual cohomological dimension; yielding some partial positive results for the Quillen conjecture in rank one. We combine these results with the known partial positive results and the known types of counterexamples to the Quillen conjecture, in order to formulate a refined variant of the conjecture.

- Alexander D. Rahm, *On the equivariant K-homology of PSL_2 of the imaginary quadratic integers*, Annales de l'Institut Fourier, 66 no. 4 (2016), pp. 1667–1689.

  We establish formulas for the part due to torsion of the equivariant K-homology of all the Bianchi groups (PSL_2 of the imaginary quadratic integers), in terms of elementary number-theoretic quantities. To achieve this, we introduce a novel technique in the computation of Bredon homology: *representation ring splitting*, which allows us to adapt the recent technique of torsion subcomplex reduction from group homology to Bredon homology.

Preprint versions of the above papers, the latest ones incorporating the referees' suggestions, are linked from the author's homepage:

`http://www.maths.nuigalway.ie/~rahm/`

Also, this page contains links to the official electronic versions of the publishers, in the cases where the author is aware of their availability.

3. A closer glance at the techniques

We only provide the core of the technique, Section 3.1, with its proofs, and refer to the published papers for the proofs in the subsequent subsections.

3.1. Reduction of torsion subcomplexes. In this section we present the ℓ-torsion subcomplexes theory of [**45**]. Let ℓ be a prime number. We require any discrete group Γ under our study to be provided with what we will call a *polytopal* Γ-*cell complex*, that is, a finite-dimensional simplicial complex X with cellular Γ-action such that each cell stabiliser fixes its cell point-wise. In practice, we relax the simplicial condition to a polyhedral one, merging finitely many simplices to a suitable polytope. We could obtain the simplicial complex back as a triangulation. We further require that the fixed point set X^G be acyclic for every non-trivial finite ℓ-subgroup G of Γ.

Then, the Γ-equivariant Farrell cohomology $\widehat{\mathrm{H}}^*_\Gamma(X;\, M)$ of X, for any trivial Γ-module M of coefficients, gives us the ℓ-primary part $\widehat{\mathrm{H}}^*(\Gamma;\, M)_{(\ell)}$ of the Farrell cohomology of Γ, as follows.

Proposition 12 (Brown [**11**])**.** *For a* Γ*-action on* X *as specified above, the canonical map*

$$\widehat{\mathrm{H}}^*(\Gamma;\, M)_{(\ell)} \to \widehat{\mathrm{H}}^*_\Gamma(X;\, M)_{(\ell)}$$

is an isomorphism.

The classical choice [**11**] is to take for X the geometric realization of the partially ordered set of non-trivial finite subgroups (respectively, non-trivial elementary Abelian ℓ-subgroups) of Γ, the latter acting by conjugation. The stabilisers are then the normalizers, which in many discrete groups are infinite. In addition, there are often great computational challenges to determine a group presentation for the normalizers. When we want to compute the module $\widehat{\mathrm{H}}^*_\Gamma(X;\, M)_{(\ell)}$ subject to Proposition 12, at least we must know the (ℓ-primary part of the) Farrell cohomology of these normalizers. The Bianchi groups are an instance where different isomorphism types can occur for this cohomology at different conjugacy classes of elementary Abelian ℓ-subgroups, both for $\ell = 2$ and $\ell = 3$. As the only non-trivial elementary Abelian 3-subgroups in the Bianchi groups are of rank 1, the orbit space ${}_\Gamma\backslash X$ consists only of one point for each conjugacy class of type $\mathbb{Z}/3$ and a corollary [**11**] from Proposition 12 decomposes the 3-primary part of the Farrell cohomology of the Bianchi groups into the direct product over their normalizers. However, due to the different possible homological types of the normalizers (in fact, two of them occur), the final result remains unclear and subject to tedious case-by-case computations of the normalizers.

In contrast, in the cell complex we are going to construct (specified in Definition 16 below), the connected components of the orbit space are for the 3-torsion in the Bianchi groups not simple points, but have either the shape $\bullet\!\!-\!\!\bullet$ or $\circ\!\!\bullet$. This dichotomy already contains the information about the occurring normalizer.

The starting point for our construction is the following definition.

Definition 13. Let ℓ be a prime number. The *ℓ-torsion subcomplex* of a polytopal Γ-cell complex X consists of all the cells of X whose stabilisers in Γ contain elements of order ℓ.

We are from now on going to require the cell complex X to admit only finite stabilisers in Γ, and we require the action of Γ on the coefficient module M to be trivial. Then obviously only cells from the *ℓ-torsion subcomplex* contribute to $\widehat{\mathrm{H}}^*_\Gamma(X;\, M)_{(\ell)}$.

Corollary 14 (Corollary to Proposition 12)**.** *There is an isomorphism between the ℓ-primary parts of the Farrell cohomology of* Γ *and the* Γ*-equivariant Farrell cohomology of the ℓ-torsion subcomplex.*

We are going to reduce the *ℓ-torsion subcomplex* to one which still carries the Γ-equivariant Farrell cohomology of X, but which can also have considerably fewer orbits of cells. This can be easier to handle in practice, and, for certain classes of groups, leads us to an explicit structural description of the Farrell cohomology of Γ. The pivotal property of this reduced ℓ-torsion subcomplex will be given in Theorem 17. Our reduction process uses the following conditions, which are imposed to a triple (σ, τ_1, τ_2) of cells in the ℓ-torsion subcomplex, where σ is a cell of dimension $n-1$, lying in the boundary of precisely the two n-cells τ_1 and τ_2, the latter cells representing two different orbits.

Condition A. The triple (σ, τ_1, τ_2) is said to satisfy Condition A if no higher-dimensional cells of the ℓ-torsion subcomplex touch σ; and if the n-cell stabilisers admit an isomorphism $\Gamma_{\tau_1} \cong \Gamma_{\tau_2}$.

Where this condition is fulfilled in the ℓ-torsion subcomplex, we merge the cells τ_1 and τ_2 along σ and do so for their entire orbits, if and only if they meet the following additional condition, that we never merge two cells the interior of which contains two points on the same orbit. We will refer by *mod ℓ cohomology* to group cohomology with $\mathbb{Z}/\ell$-coefficients under the trivial action.

Condition B. With the notation above Condition A, the inclusion $\Gamma_{\tau_1} \subset \Gamma_\sigma$ induces an isomorphism on mod ℓ cohomology.

Lemma 15 ([**45**])**.** *Let $\widetilde{X_{(\ell)}}$ be the Γ-complex obtained by orbit-wise merging two n-cells of the ℓ-torsion subcomplex $X_{(\ell)}$ which satisfy Conditions A and B. Then,*

$$\widehat{\mathrm{H}}^*_\Gamma(\widetilde{X_{(\ell)}};\, M)_{(\ell)} \cong \widehat{\mathrm{H}}^*_\Gamma(X_{(\ell)};\, M)_{(\ell)}.$$

Proof. Consider the equivariant spectral sequence in Farrell cohomology [**11**]. On the ℓ-torsion subcomplex, it includes a map

$$\widehat{\mathrm{H}}^*(\Gamma_\sigma;\, M)_{(\ell)} \xrightarrow[x \mapsto (\phi_1(x),\, \phi_2(x))]{d_1^{(n-1),*}|_{\widehat{\mathrm{H}}^*(\Gamma_\sigma;\, M)_{(\ell)}}} \widehat{\mathrm{H}}^*(\Gamma_{\tau_1};\, M)_{(\ell)} \oplus \widehat{\mathrm{H}}^*(\Gamma_{\tau_2};\, M)_{(\ell)}\ ,$$

which is the diagonal map with blocks the isomorphisms

$$\phi_i :\ \widehat{\mathrm{H}}^*(\Gamma_\sigma;\, M)_{(\ell)} \xrightarrow{\cong} \widehat{\mathrm{H}}^*(\Gamma_{\tau_i};\, M)_{(\ell)}\ ,$$

induced by the inclusions $\Gamma_{\tau_i} \hookrightarrow \Gamma_\sigma$. The latter inclusions are required to induce isomorphisms in Condition B. If for the orbit of τ_1 or τ_2 we have chosen a representative which is not adjacent to σ, then this isomorphism is composed with the isomorphism induced by conjugation with the element of Γ carrying the cell to one adjacent to σ. Hence, the map $d_1^{(n-1),*}|_{\widehat{\mathrm{H}}^*(\Gamma_\sigma;\, M)_{(\ell)}}$ has vanishing kernel, and dividing its image out of $\widehat{\mathrm{H}}^*(\Gamma_{\tau_1};\, M)_{(\ell)} \oplus \widehat{\mathrm{H}}^*(\Gamma_{\tau_2};\, M)_{(\ell)}$ gives us the ℓ-primary part $\widehat{\mathrm{H}}^*(\Gamma_{\tau_1 \cup \tau_2};\, M)_{(\ell)}$ of the Farrell cohomology of the union $\tau_1 \cup \tau_2$ of the two n-cells, once that we make use of the isomorphism $\Gamma_{\tau_1} \cong \Gamma_{\tau_2}$ of Condition A. As by Condition A no higher-dimensional cells are touching σ, higher degree differentials do not affect the result. $\square$

By a "terminal $(n-1)$-cell", we will denote an $(n-1)$-cell σ with

- modulo Γ precisely one adjacent n-cell τ,
- and such that τ has no further cells on the Γ-orbit of σ in its boundary;
- furthermore there shall be no higher-dimensional cells adjacent to σ.

And by "cutting off" the n-cell τ, we will mean that we remove τ together with σ from our cell complex.

Definition 16. A *reduced ℓ-torsion subcomplex* associated to a polytopal Γ-cell complex X is a cell complex obtained by recursively merging orbit-wise all the pairs of cells satisfying conditions A and B, and cutting off n-cells that admit a terminal $(n-1)$-cell when condition B is satisfied.

A priori, this process yields a unique reduced ℓ-torsion subcomplex only up to suitable isomorphisms, so we do not speak of "the" reduced ℓ-torsion subcomplex. The following theorem makes sure that the Γ-equivariant mod ℓ Farrell cohomology is not affected by this issue.

Theorem 17 ([**45**])**.** *There is an isomorphism between the ℓ-primary part of the Farrell cohomology of Γ and the Γ-equivariant Farrell cohomology of a reduced ℓ-torsion subcomplex obtained from X as specified above.*

Proof. We apply Proposition 12 to the cell complex X, and then we apply Lemma 15 each time that we orbit-wise merge a pair of cells of the ℓ-torsion subcomplex, or that we cut off an n-cell. □

In order to have a practical criterion for checking Condition B, we make use of the following stronger condition.

Here, we write $\mathrm{N}_{\Gamma_\sigma}$ for taking the normalizer in Γ_σ and Sylow_ℓ for picking an arbitrary Sylow ℓ-subgroup. This is well defined because all Sylow ℓ-subgroups are conjugate. We use Zassenhaus's notion for a finite group to be *ℓ-normal*, if the center of one of its Sylow ℓ-subgroups is the center of every Sylow ℓ-subgroup in which it is contained.

Condition B'. With the notation of Condition A, the group Γ_σ admits a (possibly trivial) normal subgroup T_σ with trivial mod ℓ cohomology and with quotient group G_σ; and the group Γ_{τ_1} admits a (possibly trivial) normal subgroup T_τ with trivial mod ℓ cohomology and with quotient group G_τ making the sequences

$$1 \to T_\sigma \to \Gamma_\sigma \to G_\sigma \to 1 \text{ and } 1 \to T_\tau \to \Gamma_{\tau_1} \to G_\tau \to 1$$

exact and satisfying one of the following.

(1) Either $G_\tau \cong G_\sigma$, or
(2) G_σ is ℓ-normal and $G_\tau \cong \mathrm{N}_{G_\sigma}(\mathrm{center}(\mathrm{Sylow}_\ell(G_\sigma)))$, or
(3) both G_σ and G_τ are ℓ-normal and there is a (possibly trivial) group T with trivial mod ℓ cohomology making the sequence

$$1 \to T \to \mathrm{N}_{G_\sigma}(\mathrm{center}(\mathrm{Sylow}_\ell(G_\sigma))) \to \mathrm{N}_{G_\tau}(\mathrm{center}(\mathrm{Sylow}_\ell(G_\tau))) \to 1$$

exact.

Lemma 18. *Condition B' implies Condition B.*

For the proof of (B'(2) $\Rightarrow$ B), we use Swan's extension [**65**, final corollary] to Farrell cohomology of the Second Theorem of Grün [**24**, Satz 5].

Theorem 19 (Swan)**.** *Let G be a ℓ-normal finite group, and let N be the normalizer of the center of a Sylow ℓ-subgroup of G. Let M be any trivial G-module. Then the inclusion and transfer maps both are isomorphisms between the ℓ-primary components of $\widehat{\mathrm{H}}^*(G;\, M)$ and $\widehat{\mathrm{H}}^*(N;\, M)$.*

For the proof of (B'(3) $\Rightarrow$ B), we make use of the following direct consequence of the Lyndon–Hochschild–Serre spectral sequence.

Lemma 20 ([**45**])**.** *Let T be a group with trivial mod ℓ cohomology, and consider any group extension*

$$1 \to T \to E \to Q \to 1.$$

Then the map $E \to Q$ induces an isomorphism on mod ℓ cohomology.

This statement may look like a triviality, but it becomes wrong as soon as we exchange the rôles of T and Q in the group extension. In degrees 1 and 2, our claim follows from [**11**, VII.(6.4)]. In arbitrary degree, it is more or less known and we can proceed through the following easy steps.

Proof. Consider the Lyndon–Hochschild–Serre spectral sequence associated to the group extension, namely

$$E^2_{p,q} = \mathrm{H}_p(Q;\, \mathrm{H}_q(T;\, \mathbb{Z}/\ell)) \text{ converges to } \mathrm{H}_{p+q}(E;\, \mathbb{Z}/\ell).$$

By our assumption, $\mathrm{H}_q(T;\, \mathbb{Z}/\ell)$ is trivial, so this spectral sequence concentrates in the row $q = 0$, degenerates on the second page and yields isomorphisms

$$\mathrm{H}_p(Q;\, \mathrm{H}_0(T;\, \mathbb{Z}/\ell)) \cong \mathrm{H}_p(E;\, \mathbb{Z}/\ell). \tag{1}$$

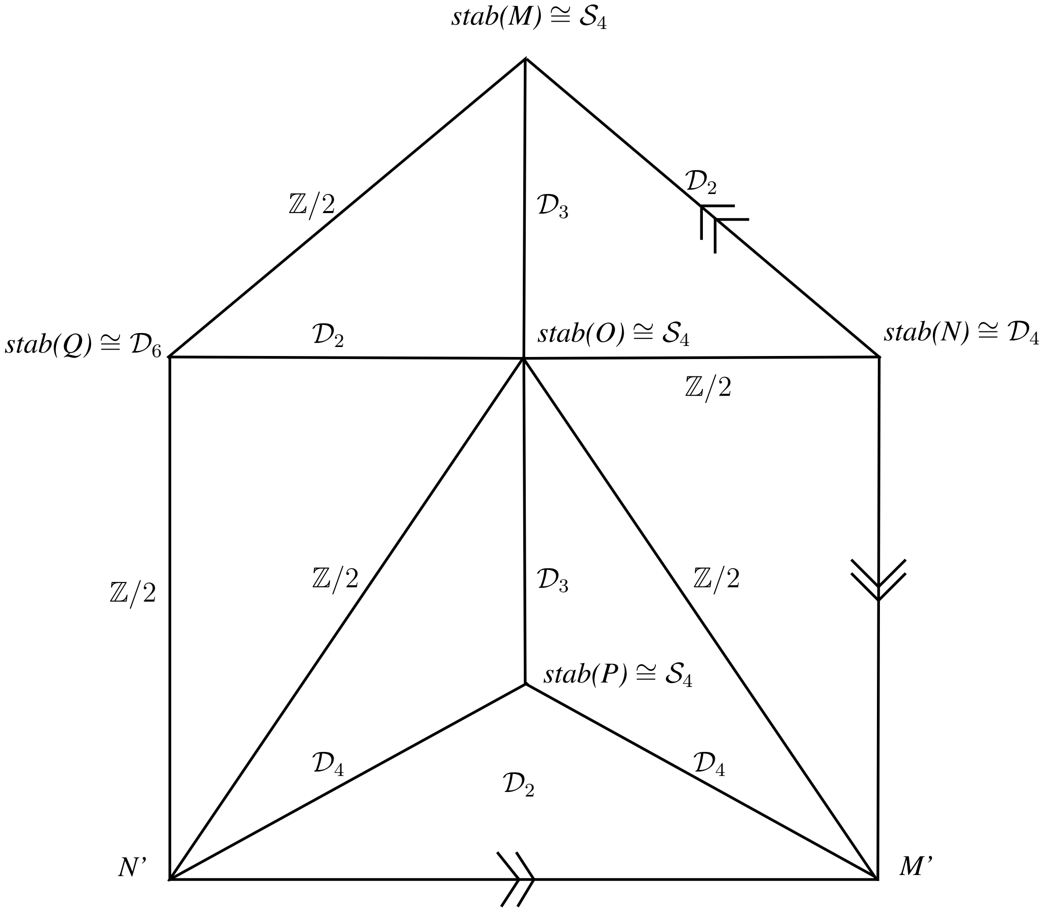
As for the modules of co-invariants, we have $((\mathbb{Z}/\ell)_T)_Q \cong (\mathbb{Z}/\ell)_E$ (see for instance [**34**]), the trivial actions of E and T induce that also the action of Q on the coefficients in $\mathrm{H}_0(T;\, \mathbb{Z}/\ell)$ is trivial. Thus, Isomorphism (1) becomes $\mathrm{H}_p(Q;\, \mathbb{Z}/\ell) \cong \mathrm{H}_p(E;\, \mathbb{Z}/\ell)$. □

The above lemma directly implies that any extension of two groups both having trivial mod ℓ cohomology, again has trivial mod ℓ cohomology.

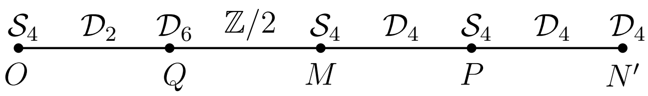
Proof of Lemma *18.* We combine Theorem 19 and Lemma 20 in the obvious way. □

Remark 21. The computer implementation [**44**] checks Conditions $B'(1)$ and $B'(2)$ for each pair of cell stabilisers, using a presentation of the latter in terms of matrices, permutation cycles or generators and relators. In the below examples however, we do avoid this case-by-case computation by a general determination of the isomorphism types of pairs of cell stabilisers for which group inclusion induces an isomorphism on mod ℓ cohomology. The latter method is the procedure of preference, because it allows us to deduce statements that hold for the entire class of groups in question.

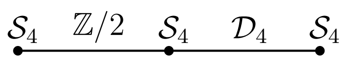
3.1.1. *Example: A 2-torsion subcomplex for $SL_3(\mathbb{Z})$.* The 2-torsion subcomplex of the cell complex described by Soulé [**64**], obtained from the action of $\mathrm{SL}_3(\mathbb{Z})$ on its symmetric space, has the following homeomorphic image.



Here, the three edges NM, NM' and $N'M'$ have to be identified as indicated by the arrows. All of the seven triangles belong with their interior to the 2-torsion subcomplex, each with stabiliser $\mathbb{Z}/2$, except for the one which is marked to have stabiliser $\mathcal{D}_2$. Using the methods described in Section 3.1, we reduce this subcomplex to



and then to



which is the geometric realization of Soulé's diagram of cell stabilisers. This yields the mod 2 Farrell cohomology as specified in [**64**].

3.1.2. *Example: Farrell cohomology of the Bianchi modular groups.* Consider the SL_2 matrix groups over the ring $\mathcal{O}_{-m}$ of integers in the imaginary quadratic number field $\mathbb{Q}(\sqrt{-m})$, with m a square-free positive integer. These groups, as well as their central quotients $\mathrm{PSL}_2(\mathcal{O}_{-m})$, are known as *Bianchi (modular) groups.* We recall the following information from [**45**] on the ℓ-torsion subcomplex of $\mathrm{PSL}_2(\mathcal{O}_{-m})$. Let Γ be a finite index subgroup in $\mathrm{PSL}_2(\mathcal{O}_{-m})$. Then any element of Γ fixing a point inside hyperbolic 3-space $\mathcal{H}$ acts as a rotation of finite order. By Felix Klein's work, we know conversely that any torsion element α is elliptic and hence fixes some geodesic line. We call this line *the rotation axis of* α. Every torsion element acts as the stabiliser of a line conjugate to one passing through the Bianchi fundamental polyhedron. We obtain the *refined cellular complex* from the action of Γ on $\mathcal{H}$ as described in [**46**], namely we subdivide $\mathcal{H}$ until the stabiliser in Γ of any cell σ fixes σ point-wise. We achieve this by computing Bianchi's fundamental polyhedron for the action of Γ, taking as a preliminary set of 2-cells its facets lying on the Euclidean hemispheres and vertical planes of the upper-half space model for $\mathcal{H}$, and then subdividing along the rotation axes of the elements of Γ.

It is well-known [**61**] that if γ is an element of finite order n in a Bianchi group, then n must be 1, 2, 3, 4 or 6, because γ has eigenvalues ρ and $\overline{\rho}$, with ρ a primitive n-th root of unity, and the trace of γ is $\rho + \overline{\rho} \in \mathcal{O}_{-m} \cap \mathbb{R} = \mathbb{Z}$. When ℓ is one of the two occurring prime numbers 2 and 3, the orbit space of this subcomplex is a graph, because the cells of dimension greater than 1 are trivially stabilized in the refined cellular complex. We can see that this graph is finite either from the finiteness of the Bianchi fundamental polyhedron, or from studying conjugacy classes of finite subgroups as in [**29**].

As in [**53**], we make use of a 2-dimensional deformation retract X of the refined cellular complex, equivariant with respect to a Bianchi group Γ. This retract has a cell structure in which each cell stabiliser fixes its cell pointwise. Since X is a deformation retract of $\mathcal{H}$ and hence acyclic,

$$\mathrm{H}^*_\Gamma(X) \cong \mathrm{H}^*_\Gamma(\mathcal{H}) \cong \mathrm{H}^*(\Gamma).$$

In Theorem 22, proven in [**45**], we give a formula expressing precisely how the Farrell cohomology of a Bianchi group with units $\{\pm 1\}$ (i.e., just excluding the Gaussian and the Eisentein integers as imaginary quadratic rings, see Section 2.4.4) depends on the numbers of conjugacy classes of non-trivial finite subgroups of the occurring five types specified in Table 1. The main step in order to prove this, is to read off the Farrell cohomology from the quotient of the reduced torsion sub-complexes.

Subgroup type	$\mathbb{Z}/2$	$\mathbb{Z}/3$	$\mathcal{D}_2$	$\mathcal{D}_3$	$\mathcal{A}_4$
Number of conjugacy classes	λ_4	λ_6	μ_2	μ_3	μ_T

TABLE 1. The non-trivial finite subgroups of $\mathrm{PSL}_2(\mathcal{O}_{-m})$ have been classified by Klein [**27**]. Here, $\mathbb{Z}/n$ is the cyclic group of order n, the dihedral groups are $\mathcal{D}_2$ with four elements and $\mathcal{D}_3$ with six elements, and the tetrahedral group is isomorphic to the alternating group $\mathcal{A}_4$ on four letters. Formulas for the numbers of conjugacy classes counted by the Greek symbols have been given by Krämer [**29**].

Krämer's formulas [**29**] express the numbers of conjugacy classes of the five types of non-trivial finite subgroups given in Table 1. We are going to use the symbols of that table also for the numbers of conjugacy classes in Γ, where Γ is a finite index subgroup in a Bianchi group. Recall that for $\ell = 2$ and $\ell = 3$, we can express the the dimensions of the homology of Γ with coefficients in the field $\mathbb{F}_\ell$ with ℓ elements in degrees above the virtual cohomological dimension of the Bianchi groups – which is 2 – by the Poincaré series

$$P^\ell_\Gamma(t) := \sum_{q>2}^{\infty} \dim_{\mathbb{F}_\ell} \mathrm{H}_q(\Gamma;\, \mathbb{F}_\ell)\, t^q,$$

which has been suggested by Grunewald. Further let $P_{\bigcirc}(t) := \frac{-2t^3}{t-1}$, which equals the Poincaré series $P^2_\Gamma(t)$ of the groups Γ the quotient of the reduced 2–torsion sub-complex of which is a circle. Denote by

- $P^*_{\mathcal{D}_2}(t) := \frac{-t^3(3t-5)}{2(t-1)^2}$, the Poincaré series over

$$\dim_{\mathbb{F}_2} \mathrm{H}_q(\mathcal{D}_2;\, \mathbb{F}_2) - \frac{3}{2} \dim_{\mathbb{F}_2} \mathrm{H}_q(\mathbb{Z}/2;\, \mathbb{F}_2)$$

- and by $P^*_{\mathcal{A}_4}(t) := \frac{-t^3(t^3-2t^2+2t-3)}{2(t-1)^2(t^2+t+1)}$, the Poincaré series over

$$\dim_{\mathbb{F}_2} \mathrm{H}_q(\mathcal{A}_4;\, \mathbb{F}_2) - \frac{1}{2} \dim_{\mathbb{F}_2} \mathrm{H}_q(\mathbb{Z}/2;\, \mathbb{F}_2).$$

In 3-torsion, let $P_{\bullet\!-\!\bullet}(t) := \frac{-t^3(t^2-t+2)}{(t-1)(t^2+1)}$, which equals the Poincaré series $P^3_\Gamma(t)$ for those Bianchi groups, the quotient of the reduced 3–torsion sub-complex of which is a single edge without identifications.

Theorem 22. *For any finite index subgroup* Γ *in a Bianchi group with units* $\{\pm 1\}$*, the group homology in degrees above its virtual cohomological dimension is given by the Poincaré series*

$$P_\Gamma^2(t) = \left(\lambda_4 - \frac{3\mu_2 - 2\mu_T}{2}\right) P_{\circ\!\bullet}(t) + (\mu_2 - \mu_T) P^*_{\mathcal{D}_2}(t) + \mu_T P^*_{\mathcal{A}_4}(t)$$

and

$$P_\Gamma^3(t) = \left(\lambda_6 - \frac{\mu_3}{2}\right) P_{\circ\!\bullet}(t) + \frac{\mu_3}{2} P_{\bullet\!-\!\bullet}(t).$$

More general results are stated in Section 2.4.1 above.

3.1.3. *Example: Farrell cohomology of Coxeter (tetrahedral) groups.* Recall that a Coxeter group is a group admitting a presentation

$$\langle g_1, g_2, ..., g_n \mid (g_i g_j)^{m_{i,j}} = 1 \rangle,$$

where $m_{i,i} = 1$; for $i \neq j$ we have $m_{i,j} \geq 2$; and $m_{i,j} = \infty$ is permitted, meaning that $(g_i g_j)$ is not of finite order. As the Coxeter groups admit a contractible classifying space for proper actions [**15**], their Farrell cohomology yields all of their group cohomology. So in this section, we make use of this fact to determine the latter. For facts about Coxeter groups, and especially for the Davis complex, we refer to [**15**]. Recall that the simplest example of a Coxeter group, the dihedral group $\mathcal{D}_n$, is an extension

$$1 \to \mathbb{Z}/n \to \mathcal{D}_n \to \mathbb{Z}/2 \to 1.$$

So we can make use of the original application [**67**] of Wall's lemma to obtain its mod ℓ homology for prime numbers $\ell > 2$,

$$\mathrm{H}_q(\mathcal{D}_n; \mathbb{Z}/\ell) \cong \begin{cases} \mathbb{Z}/\ell, & q = 0, \\ \mathbb{Z}/\gcd(n, \ell), & q \equiv 3 \text{ or } 4 \mod 4, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 23 ([**45**])**.** *Let* $\ell > 2$ *be a prime number. Let* Γ *be a Coxeter group admitting a Coxeter system with at most four generators, and relator orders not divisible by* ℓ^2*. Let* $Z_{(\ell)}$ *be the* ℓ*–torsion sub-complex of the Davis complex of* Γ*. If* $Z_{(\ell)}$ *is at most one-dimensional and its orbit space contains no loop or bifurcation, then the* mod ℓ *homology of* Γ *is isomorphic to* $(\mathrm{H}_q(\mathcal{D}_\ell; \mathbb{Z}/\ell))^m$*, with* m *the number of connected components of the orbit space of* $Z_{(\ell)}$*.*

The conditions of this theorem are for instance fulfilled by the Coxeter tetrahedral groups; the exponent m has been specified for each of them in the tables in [**45**]. In the easier case of Coxeter triangle groups, we can sharpen the statement as follows. The non-spherical and hence infinite *Coxeter triangle groups* are given by the presentation

$$\langle\, a, b, c \mid a^2 = b^2 = c^2 = (ab)^p = (bc)^q = (ca)^r = 1 \,\rangle\,,$$

where $2 \leq p, q, r \in \mathbb{N}$ and $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq 1$.

Proposition 24 ([**45**])**.** *For any prime number $\ell > 2$, the* mod ℓ *homology of a Coxeter triangle group is given as the direct sum over the* mod ℓ *homology of the dihedral groups $\mathcal{D}_p$, $\mathcal{D}_q$ and $\mathcal{D}_r$.*

3.2. The non-central torsion subcomplex. In the case of a trivial kernel of the action on the polytopal Γ-cell complex, torsion subcomplex reduction allows one to establish general formulas for the Farrell cohomology of Γ [**45**]. In contrast, for instance the action of $\mathrm{SL}_2(\mathcal{O}_{-m})$ on hyperbolic 3-space has the 2-torsion group $\{\pm 1\}$ in the kernel; since every cell stabiliser contains 2-torsion, the 2-torsion subcomplex does not ease our calculation in any way. We can remedy this situation by considering the following object, on whose cells we impose a supplementary property.

Definition 25. The *non-central ℓ-torsion subcomplex* of a polytopal Γ-cell complex X consists of all the cells of X whose stabilisers in Γ contain elements of order ℓ that are not in the center of Γ.

We note that this definition yields a correspondence between, on one side, the *non-central* ℓ-torsion subcomplex for a group action with kernel the center of the group, and on the other side, the ℓ-torsion subcomplex for its central quotient group. In [**8**], this correspondence has been used in order to identify the *non-central* ℓ-torsion subcomplex for the action of $\mathrm{SL}_2(\mathcal{O}_{-m})$ on hyperbolic 3-space as the ℓ-torsion subcomplex of $\mathrm{PSL}_2(\mathcal{O}_{-m})$. However, incorporating the non-central condition for $\mathrm{SL}_2(\mathcal{O}_{-m})$ introduces significant technical obstacles, which were addressed in that paper, establishing the following theorem for any finite index subgroup Γ in $\mathrm{SL}_2(\mathcal{O}_{-m})$. Denote by X a Γ-equivariant retract of $\mathrm{SL}_2(\mathbb{C})/\mathrm{SU}_2$, by X_s the 2-torsion subcomplex with respect to $\mathrm{P}\Gamma$ (the "non-central" 2-torsion subcomplex for Γ), and by X_s' the part of it with higher 2-rank. Further, let v denote the number of conjugacy classes of subgroups of higher 2-rank, and define $\mathrm{sign}(v) := \begin{cases} 0, & v = 0, \\ 1, & v > 0. \end{cases}$ For $q \in \{1, 2\}$, denote the dimension $\dim_{\mathbb{F}_2} \mathrm{H}^q({}_\Gamma\backslash X;\, \mathbb{F}_2)$ by β^q.

Theorem 26 ([**8**])**.** *The E_2 page of the equivariant spectral sequence with $\mathbb{F}_2$-coefficients associated to the action of Γ on X is concentrated in the columns $n \in \{0, 1, 2\}$ and has the following form.*

$q = 4k + 3$	$E_2^{0,3}(X_s)$	$E_2^{1,3}(X_s) \oplus (\mathbb{F}_2)^{a_1}$	$(\mathbb{F}_2)^{a_2}$
$q = 4k + 2$	$\mathrm{H}^2_\Gamma(X'_s) \oplus (\mathbb{F}_2)^{1-\mathrm{sign}(v)}$	$(\mathbb{F}_2)^{a_3}$	$\mathrm{H}^2(_\Gamma\backslash X)$
$q = 4k + 1$	$E_2^{0,1}(X_s)$	$E_2^{1,1}(X_s) \oplus (\mathbb{F}_2)^{a_1}$	$(\mathbb{F}_2)^{a_2}$
$q = 4k$	$\mathbb{F}_2$	$\mathrm{H}^1(_\Gamma\backslash X)$	$\mathrm{H}^2(_\Gamma\backslash X)$
$k \in \mathbb{N} \cup \{0\}$	$n = 0$	$n = 1$	$n = 2$

where

$$\begin{aligned} a_1 &= \chi(_\Gamma\backslash X_s) - 1 + \beta^1(_\Gamma\backslash X) + c \\ a_2 &= \beta^2(_\Gamma\backslash X) + c \\ a_3 &= \beta^1(_\Gamma\backslash X) + v - \mathrm{sign}(v). \end{aligned}$$

In order to derive the example stated in Section 2.4.3 above, we combine the latter theorem with the following determination (carried out in [**8**]) of the d_2-differentials on the four possible (cf. Table 2) connected component types , ,  and  of the reduced non-central 2-torsion subcomplex for the full SL_2 groups over the imaginary quadratic number rings.

Lemma 27 ([**8**])**.** *The d_2 differential in the equivariant spectral sequence associated to the action of $\mathrm{SL}_2(\mathcal{O}_{-m})$ on hyperbolic space is trivial on components of the non-central 2-torsion subcomplex quotient*

- *of type*  *in dimensions $q \equiv 1 \bmod 4$ if and only if it is trivial on these components in dimensions $q \equiv 3 \bmod 4$.*
- *of type* *.*
- *of types*  *and*  *in dimensions $q \equiv 3 \bmod 4$.*

3.3. Application to equivariant *K*-homology. In order to adapt torsion subcomplex reduction to Bredon homology and prove Theorem 5, we need to perform a "representation ring splitting".

Representation ring splitting. The classification of Felix Klein [**27**] of the finite subgroups in $\mathrm{PSL}_2(\mathcal{O})$ is recalled in Table 1. We further use the existence of geometric models for the Bianchi groups in which all edge stabilisers are finite cyclic and all cells of dimension 2 and higher are trivially stabilised. Therefore, the system of finite subgroups of the Bianchi groups admits inclusions only emanating from cyclic groups. This makes the Bianchi groups and their subgroups subject to the splitting of Bredon homology stated in Theorem 5.

The proof of Theorem 5 is based on the above particularities of the Bianchi groups, and applies the following splitting lemma for the involved representation rings to a Bredon complex for $(\underline{\mathrm{E}}\Gamma, \Gamma)$.

TABLE 2. Connected component types of reduced torsion subcomplex quotients for the PSL_2 Bianchi groups. The exhaustiveness of this table has been established using theorems of Krämer [**8**].

2–torsion subcomplex components	counted by	3–torsion subcomplex components	counted by
$\circ\!\!\bullet\ \mathbb{Z}/2$	$o_2 = \lambda_4 - \lambda_4^*$	$\circ\!\!\bullet\ \mathbb{Z}/3$	$o_3 = \lambda_6 - \lambda_6^*$
$\mathcal{A}_4 \bullet\!\!-\!\!\bullet \mathcal{A}_4$	ι_2	$\mathcal{D}_3 \bullet\!\!-\!\!\bullet \mathcal{D}_3$	$\iota_3 = \lambda_6^*$
$\mathcal{D}_2 \ominus \mathcal{D}_2$	θ		
$\mathcal{D}_2 \circ\!\!-\!\!\bullet\ \mathcal{A}_4$	ρ		

Lemma 28 ([**47**]). *Consider a group Γ such that every one of its finite subgroups is either cyclic of order at most 3, or of one of the types $\mathcal{D}_2, \mathcal{D}_3$ or $\mathcal{A}_4$. Then there exist bases of the complex representation rings of the finite subgroups of Γ, such that simultaneously every morphism of representation rings induced by inclusion of cyclic groups into finite subgroups of Γ, splits as a matrix into the following diagonal blocks.*

(1) A block of rank 1 induced by the trivial and regular representations,
(2) a block induced by the 2–torsion subgroups
(3) and a block induced by the 3–torsion subgroups.

As this splitting holds simultaneously for every morphism of representation rings, we have such a splitting for every morphism of formal sums of representation rings, and hence for the differential maps of the Bredon complex for any Bianchi group and any of their subgroups.

The bases that are mentioned in the above lemma, are obtained by elementary base transformations from the canonical basis of the complex representation ring of a finite group to a basis whose matrix form has

- its first row concentrated in its first entry, for a finite cyclic group (edge stabiliser). The base transformation is carried out by summing over all representations to replace the trivial representation by the regular representation.

- its first column concentrated in its first entry, for a finite non-cyclic group (vertex stabiliser). The base transformation is carried out by subtracting the trivial representation from each representation, except from itself.

The details are provided in [**47**].

3.4. Chen–Ruan orbifold cohomology of the complexified orbifolds. Let Γ be a discrete group acting *properly*, i.e. with finite stabilizers, by diffeomorphisms on a manifold Y. For any element $g \in \Gamma$, denote by $C_\Gamma(g)$ the centralizer of g in Γ. Denote by Y^g the subset of Y consisting of the fixed points of g.

Definition 29. Let $T \subset \Gamma$ be a set of representatives of the conjugacy classes of elements of finite order in Γ. Then we set

$$\mathrm{H}^*_{orb}([Y/\Gamma]) := \bigoplus_{g \in T} \mathrm{H}^* \left(Y^g / C_\Gamma(g); \mathbb{Q}\right).$$

It can be checked that this definition gives the vector space structure of the orbifold cohomology defined by Chen and Ruan [**14**], if we forget the grading of the latter. We can verify this fact using arguments analogous to those used by Fantechi and Göttsche [**18**] in the case of a finite group Γ acting on Y. The additional argument needed when considering some element g in Γ of infinite order, is the following. As the action of Γ on Y is proper, g does not admit any fixed point in Y. Thus, $\mathrm{H}^* \left(Y^g / C_\Gamma(g); \mathbb{Q}\right) = \mathrm{H}^* \left(\emptyset; \mathbb{Q}\right) = 0$.

Our main results on the vector space structure of the Chen–Ruan orbifold cohomology of Bianchi orbifolds are the below two theorems.

Theorem 30 ([**47**])**.** *For any element γ of order 3 in a finite index subgroup Γ in a Bianchi group with units $\{\pm 1\}$, the quotient space $\mathcal{H}^\gamma /_{C_\Gamma(\gamma)}$ of the rotation axis modulo the centralizer of γ is homeomorphic to a circle.*

Theorem 31 ([**47**])**.** *Let γ be an element of order 2 in a Bianchi group Γ with units $\{\pm 1\}$. Then, the homeomorphism type of the quotient space $\mathcal{H}^\gamma /_{C_\Gamma(\gamma)}$ is*

- *an edge without identifications, if $\langle \gamma \rangle$ is contained in a subgroup of type $\mathcal{D}_2$ inside Γ and*
- *a circle, otherwise.*

Denote by $\lambda_{2\ell}$ the number of conjugacy classes of subgroups of type $\mathbb{Z}/_{\ell\mathbb{Z}}$ in a finite index subgroup Γ in a Bianchi group with units $\{\pm 1\}$. Denote by $\lambda^*_{2\ell}$ the number of conjugacy classes of subgroups of type $\mathbb{Z}/_{\ell\mathbb{Z}}$ which are contained in a subgroup of type $\mathcal{D}_n$ in Γ. By [**47**], there are $2\lambda_6 - \lambda_6^*$ conjugacy classes of elements of order 3. As a result of Theorems 30 and 31, the vector space structure of the orbifold cohomology of $[\mathcal{H}^3_\mathbb{R}/\Gamma]$ is given as

$$\mathrm{H}^\bullet_{orb}([\mathcal{H}^3_\mathbb{R}/\Gamma]) \cong$$
$$\mathrm{H}^\bullet\left(\mathcal{H}_\mathbb{R}/_\Gamma;\,\mathbb{Q}\right) \bigoplus^{\lambda_4^*} \mathrm{H}^\bullet\left(\bullet\!\!-\!\!\bullet;\,\mathbb{Q}\right) \bigoplus^{(\lambda_4-\lambda_4^*)} \mathrm{H}^\bullet\left(\circ\!\!\bullet;\,\mathbb{Q}\right) \bigoplus^{(2\lambda_6-\lambda_6^*)} \mathrm{H}^\bullet\left(\circ\!\!\bullet;\,\mathbb{Q}\right).$$

The (co)homology of the quotient space $\mathcal{H}_\mathbb{R}/_\Gamma$ has been computed numerically for a large range of Bianchi groups [**66**], [**60**], [**49**]; and bounds for its Betti numbers have been given in [**30**]. Krämer [**29**] has determined number-theoretic formulas for the numbers $\lambda_{2\ell}$ and $\lambda_{2\ell}^*$ of conjugacy classes of finite subgroups in the full Bianchi groups. Krämer's formulas have been evaluated for hundreds of thousands of Bianchi groups [**45**], and these values are matching with the ones from the orbifold structure computations with [**43**] in the cases where the latter are available.

When we pass to the complexified orbifold $[\mathcal{H}^3_\mathbb{C}/\Gamma]$, the real line that is the rotation axis in $\mathcal{H}_\mathbb{R}$ of an element of finite order, becomes a complex line. However, the centralizer still acts in the same way by reflections and translations. So, the interval $\bullet\!\!-\!\!\bullet$ as a quotient of the real line yields a stripe $\bullet\!\!-\!\!\bullet \times \mathbb{R}$ as a quotient of the complex line. And the circle $\circ\!\!\bullet$ as a quotient of the real line yields a cylinder $\circ\!\!\bullet \times \mathbb{R}$ as a quotient of the complex line. Therefore, using the degree shifting numbers computed in [**47**], we obtain the result of Theorem 9,

$$\mathrm{H}^d_{orb}\left([\mathcal{H}^3_\mathbb{C}/\Gamma]\right) \cong \mathrm{H}^d\left(\mathcal{H}_\mathbb{C}/_\Gamma;\,\mathbb{Q}\right) \oplus \begin{cases} \mathbb{Q}^{\lambda_4+2\lambda_6-\lambda_6^*}, & d=2, \\ \mathbb{Q}^{\lambda_4-\lambda_4^*+2\lambda_6-\lambda_6^*}, & d=3, \\ 0, & \text{otherwise.} \end{cases}$$

4. Other achievements

(1) Alexander D. Rahm, *On a question of Serre*, Comptes Rendus Mathématique de l'Académie des Sciences - Paris (2012), presented by Jean-Pierre Serre [**41**].

Consider an imaginary quadratic number field $\mathbb{Q}(\sqrt{-m})$, with m a square-free positive integer, and its ring of integers $\mathcal{O}$. The *Bianchi groups* are the groups $\mathrm{SL}_2(\mathcal{O})$. Further consider the Borel–Serre compactification [**63**] of the quotient of hyperbolic 3–space $\mathcal{H}$ by a finite index subgroup Γ in a Bianchi group, and in particular the following question which Serre posed on page 514 of the quoted article. Consider the map α induced on homology when attaching the boundary into the Borel–Serre compactification.

How can one determine the kernel of α (in degree 1) ?

Serre used a global topological argument and obtained the rank of the kernel of α. In the quoted article, Serre did add the question what submodule precisely this kernel is. Through a local topological study, we can decompose the kernel of α into its parts associated to each cusp.

(2) Alexander D. Rahm and Mathias Fuchs, *The integral homology of* PSL_2 *of imaginary quadratic integers with non-trivial class group*, Journal of Pure and Applied Algebra (2011) [**53**].

We show that a cellular complex described by Flöge allows to determine the integral homology of the Bianchi groups $\mathrm{PSL}_2(\mathcal{O}_{-m})$. We use this to compute in the cases $m = 5, 6, 10, 13$ and 15 with non-trivial class group the integral homology of $\mathrm{PSL}_2(\mathcal{O}_{-m})$. Previously, this was only known in the cases $m = 1, 2, 3, 7$ and 11 with trivial class group.

(3) Alexander D. Rahm, *Higher torsion in the Abelianization of the full Bianchi groups*, LMS J. of Computation and Mathematics (2013) [**49**].

Denote by $\mathbb{Q}(\sqrt{-m})$, with m a square-free positive integer, an imaginary quadratic number field, and by $\mathcal{O}_{-m}$ its ring of integers. The *Bianchi groups* are the groups $\mathrm{SL}_2(\mathcal{O}_{-m})$. In the literature, there has been so far no example of p-torsion in the integral homology of the full Bianchi groups, for p a prime greater than the order of elements of finite order in the Bianchi group, which is at most 6.

However, extending the scope of the computations, we can observe examples of torsion in the integral homology of the quotient space, at prime numbers as high as for instance $p = 80737$ at the discriminant -1747.

(4) Alexander D. Rahm and Mehmet Haluk Şengün, *On Level One Cuspidal Bianchi Modular Forms*, LMS Journal of Computation and Mathematics (2013) [**51**].

In this paper, we present the outcome of extensive computer calculations, locating several of the very rare instances of level one cuspidal Bianchi modular forms that are not lifts of elliptic modular forms.

(5) Alexander D. Rahm, *The subgroup measuring the defect of the Abelianization of* $\mathrm{SL}_2(\mathbb{Z}[i])$, Journal of Homotopy and Related Structures (2014) [**57**].

There is a natural inclusion of $\mathrm{SL}_2(\mathbb{Z})$ into $\mathrm{SL}_2(\mathbb{Z}[i])$, but it does not induce an injection of commutator factor groups (Abelianizations).
In order to see where and how the 3-torsion of the Abelianization of $\mathrm{SL}_2(\mathbb{Z})$ disappears, we study a double cover of the amalgamated product decomposition $\mathrm{SL}_2(\mathbb{Z}) \cong (\mathbb{Z}/4\mathbb{Z}) *_{(\mathbb{Z}/2\mathbb{Z})} (\mathbb{Z}/6\mathbb{Z})$ inside $\mathrm{SL}_2(\mathbb{Z}[i])$;
and then compute the homology of the covering amalgam.

(6) Alexander D. Rahm, *Complexifiable characteristic classes*, Journal of Homotopy & Related Structures (2015) [**56**].

We examine the topological characteristic cohomology classes of complexified vector bundles. In particular, all the classes coming from the real vector bundles underlying the complexification are determined.

Concerning Item 4 above, we shall now review some details.

Bianchi modular forms are automorphic forms over an imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$, of cohomological type, associated to a Bianchi group. Even though modern studies of Bianchi modular forms go back to the mid 1960's, most of the fundamental problems surrounding their theory are still wide open. In the paper [**51**], we report on our extensive computations that show the paucity of "genuine" level one cuspidal Bianchi modular forms.

Let $S_k(1)$ denote the space of level one weight $k+2$ cuspidal Bianchi modular forms over $\mathbb{Q}(\sqrt{-d})$. In their 2010 paper [**22**], Finis, Grunewald and Tirao computed the dimension of the subspace $L_k(1)$ of $S_k(1)$ which is formed by (twists of) those forms which arise from elliptic cuspidal modular forms via base-change or arise from a quadratic extension of $\mathbb{Q}(\sqrt{-d})$ via automorphic induction (see [**22**] for these notions). The orthogonal complement to $L_k(1)$ in $S_k(1)$ is called the space of *genuine* modular forms, and is investigated numerically due to the conjectural connections between the spaces $S_0(1)$ and Abelian varieties defined over $\mathbb{Q}(\sqrt{-d})$ of GL_2-type. There have been previous reports, however of limited size, in the 2009 paper [**13**] of Calegari and Mazur (the computations in this

$\|D\|$	**7**	**11**	**71**	**87**	**91**	**155**	**199**	**223**	**231**	**339**	**344**
k	12	10	1	2	6	4	1	0	4	1	1
dim	2	2	2	2	2	2	4	2	2	2	2

$\|D\|$	**407**	**415**	**455**	**483**	**571**	**571**	**643**	**760**	**1003**	**1003**	**1051**
k	0	0	0	1	0	1	0	2	0	1	0
dim	2	2	2	2	2	2	2	2	2	2	2

TABLE 3. The cases where there are genuine classes

paper were carried out by Pollack and Stein) and in the 2010 paper [**22**] of Finis, Grunewald and Tirao. While the computations in [**13**] were limited to the case $d = 2$, the computations in [**22**] covered ten imaginary quadratic fields.

It was observed in [**13**] that for $2k \leq 96$, one has $L_{2k}(1) = S_{2k}(1)$. The computations of [**22**] extended those of [**13**]. An interesting outcome of the data collected in [**22**] is that except in two of the 946 spaces they computed, one has $L_k(1) = S_k(1)$. The exceptional cases are $(d, k) = (7, 12)$ and $(d, k) = (11, 10)$. In both cases, there is a two-dimensional complement to $L_k(1)$ inside $S_k(1)$.

Using a different and more efficient approach, we computed, over more than 800 processor-days, the dimension of 4986 different spaces $S_k(1)$ over 186 different imaginary quadratic fields. The precise scope of our computations is given in [**51**]. In only 22 of these spaces were we able to observe genuine forms. The precise data about these exceptional cases is provided in Table 3. We note that in [**51**], some further subspaces are tabulated, which are in fact populated by CM-forms (arising through automorphic induction).

As usual, the starting point of our approach is the so called "Eichler-Shimura-Harder" isomorphism which allows us to replace $S_k(1)$ with the cohomology of the relevant Bianchi group with special non-trivial coefficients. Then to compute this cohomology space, we use the program *Bianchi.gp* [**43**], which analyzes the structure of the Bianchi group via its action on hyperbolic 3-space (which is isomorphic to the associated symmetric space $\mathrm{SL}_2(\mathbb{C})/\mathrm{SU}_2$). We then feed this group-geometric information into an equivariant spectral sequence that gives us an explicit description of the second cohomology of the Bianchi group, with the relevant coefficients.

These investigations are currently being extended to higher levels, in joint work with M. Haluk Şengün and Panagiotis Tsaknias.

5. Future work on torsion in the homology of discrete groups

Objectives. For each of the following objectives, a state of the art description is provided above (Section 2.4.m for Objective 5.m, where m runs from 1 to 5).

5.1. Extension of the technique for higher rank matrix groups.

The results so far obtained for linear groups mainly concern rank 2 matrix groups. Some technical difficulties await us in treating higher rank matrix groups with my technique. I want to overcome these difficulties, and establish formulas for the Farrell–Tate cohomology of PSL_n and PGL_n, $n \geq 3$, over rings of integers in number fields.

While I am going to focus on PSL_n and PGL_n, I am going to keep the more general picture of reductive groups in mind, in the hope that along the way, I can lay the foundations for adaptations of torsion subcomplex reduction in this direction.

Cohomology of the Hilbert modular groups As a stepping-stone for reaching Objective 5.1, the cohomology of a collection of Hilbert modular groups (SL_2 over totally real quadratic integers) shall be computed explicitly, because Hilbert modular groups occur as block subgroups in the higher rank matrix groups to be studied.

5.2. Investigation of the refined Quillen conjecture.

One of the applications of the rank filtration methods for higher rank arithmetic groups G is checking the refined Quillen conjecture (stated as Conjecture 3 above) on SL_3 over number fields, using the formulas to be established as part of Objective 5.1. My goal is to pursue this towards a final refinement of the Quillen conjecture (in joint work with Matthias Wendt).

I will also consider possible extensions of the Quillen conjecture in several directions. It is possible to ask versions of Quillen's conjecture for reductive groups G other than GL_n or SL_n. In such a formulation, I want to know if the cohomology ring $\mathrm{H}^\bullet(G(\mathcal{O}_{K,S}); \mathbb{F}_\ell)$ is free over the topological cohomology ring $\mathrm{H}^\bullet_{\mathrm{cts}}(G(\mathbb{C}); \mathbb{F}_\ell)$.

5.3. Adaptation of the technique to groups with non-trivial centre.

My technique of torsion subcomplex reduction, which originally had been designed for groups with trivial centre needs to be adapted to treat also groups with non-trivial centre. This yields technical difficulties, because the torsion subcomplexes are in the latter case no longer automatically proper subcomplexes.

5.4. Application to equivariant K-homology. The technique of torsion subcomplex reduction will be adapted from group homology to Bredon homology with coefficients in the complex representation rings, and with respect to the family of finite subgroups. This will be used to obtain formulas for this Bredon homology, and by the Atiyah–Hirzebruch spectral sequence, formulas for equivariant K-homology of the investigated arithmetic groups. Equivariant K-homology is the geometric-topological side of the Baum–Connes conjecture: Baum and Connes constructed a homomorphism from the equivariant K-homology to the K-theory of the reduced C^*-algebras of a given group called the assembly map. The Baum–Connes conjecture states that the assembly map is an isomorphism for all finitely presented groups; it implies several important conjectures in topology, geometry, algebra and functional analysis: Groups for which the assembly map is surjective satisfy the Kaplansky–Kadison conjecture on the idempotents; groups for which it is injective, satisfy the strong Novikov conjecture and the direction of the Gromov–Lawson–Rosenberg conjecture predicting the vanishing of the higher $\hat{A}$-genera.

5.5. Chen–Ruan orbifold cohomology of the complexified orbifolds. I want to establish formulas for the twisted sector part of the Chen–Ruan orbifold cohomology of complexifications of the orbifolds given by the action of the arithmetic groups studied for Objective 5.1 on their symmetric space. Ruan's crepant resolution conjecture is still open on higher-dimensional orbifolds that are not global quotients, and once that I know the Chen–Ruan orbifold cohomology of these complexified orbifolds explicitly, I will examine Ruan's conjecture on them together with my collaborator Fabio Perroni.

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