DIMENSION COMPUTATIONS FOR CUSPIDAL BIANCHI MODULAR FORMS – AN ALGORITHM

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Abstract. The second author has conceived and implemented an algorithm for the computation of the dimension of the spaces of cuspidal Bianchi modular forms, for varying discriminant, congruence subgroup level and coefficient weight. We state it here.

The setting of the below algorithms is specified in [RS12].

Algorithm 1 Computing the action on the weight module $\mathrm{Sym}^k(\mathbb{C}^2) \otimes \mathrm{Sym}^k(\overline{\mathbb{C}^2})$

Input: A weight $k \in \mathbb{N} \cup \{0\}$. A matrix $B \in \mathrm{SL}_2(\mathbb{C})$.
Output: A matrix specifying the action of the element $B \in \mathrm{SL}_2(\mathbb{C})$ on the weight module $\mathrm{Sym}^k(\mathbb{C}^2) \otimes \mathrm{Sym}^k(\overline{\mathbb{C}^2})$.

Let $\overline{B}$ be the complex conjugate of B.
Let $\mathbb{C}[x, y]$ be the polynomial ring over $\mathbb{C}$ on two variables x and y.
The output is $\mathrm{Symm}_k(B) \otimes \mathrm{Symm}_k(\overline{B})$,
where the $(k + 1) \times (k + 1)$–matrix $\mathrm{Symm}_k(B)$ is computed as follows.
for $i \in \{0, 1, \dots, k\}$, **do**
  Let $Q := (B_{1,1}x + B_{1,2}y)^{k-i}(B_{2,1}x + B_{2,2}y)^i$.
  for $j \in \{0, 1, \dots, k\}$, **do**
    Set $\mathrm{Symm}_k(B)_{i+1,j+1}$ to be the coefficient in $\mathcal{O}_{-m}$,
    with which the monomial $x^{(k-j)}y^j$ occurs in the polynomial Q.
  end for
end for
The remaining entries of $\mathrm{Symm}_k(B)$ are zeroes.
Return $\mathrm{Symm}_k(B) \otimes \mathrm{Symm}_k(\overline{B})$.

We will refer to the Eckmann–Shapiro Lemma as "Shapiro's Lemma".

References

[RS12] Alexander D. Rahm and Mehmet Haluk Şengün, *On Level One Cuspidal Bianchi Modular Forms*, appeared in LMS J. Comp. & Math., `http://hal.archives-ouvertes.fr/hal-00589184/en/`

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Algorithm 2 Computing the action on the induced module of Shapiro's Lemma

Input: A field $K := \mathbb{Q}(\sqrt{-m})$, with m a square-free positive integer,
its ring of integers $\mathcal{O}_{-m}$, and an ideal J in this ring.
A matrix $B \in \mathrm{SL}_2(\mathcal{O}_{-m})$.
Output: A matrix specifying the action of the element $B \in \mathrm{SL}_2(\mathcal{O}_{-m})$
on the projective line over the ring $\mathcal{O}_{-m}/_J$.

From $\mathcal{O}_{-m}$ and J, precompute using Steve Donnelly's MAGMA function `ProjectiveLine`:
• a set S of pairs of elements of $\mathcal{O}_{-m}$ which, when reduced, constitute representatives over $\mathcal{O}_{-m}/_J$ of the elements of projective line $\mathbb{P}^1(\mathcal{O}_{-m}/_J)$.
• a function r which inputs any pair (x, y) of elements of $\mathcal{O}_{-m}$ and maps it to a pair in the set S which represents the class of (x, y) in $\mathbb{P}^1(\mathcal{O}_{-m}/_J)$.

for $i \in \{1, \dots,\text{cardinality}(S)\}$ **do**
  Apply the matrix B to the i-th pair of S, and denote the result by (x, y).
  Find the index n of (x, y) in S.
  Append n to a sequence N.
end for
Now, N is a permutation of the set $\{1, \dots,\text{cardinality}(S)\}$.
Find a permutation matrix P over K which transforms $\{1, \dots,\text{cardinality}(S)\}$ into N.
Return P.

Algorithm 3 Computing the dimension of the space of cusp forms

Input: A field $K := \mathbb{Q}(\sqrt{-m})$, with m a square-free positive integer,
its ring of integers $\mathcal{O}_{-m}$, and an ideal J in this ring. A weight $k \in \mathbb{N} \cup \{0\}$.
Edge stabilisers $E_1, \ldots, E_\ell$ with differential matrices $d_1, \ldots, d_\ell$;
edge identifiers $g_1, \ldots, g_n \in \mathrm{SL}_2(\mathcal{O}_{-m})$;
the number N_D of 2–cells in the orbit space $\mathcal{H}/_{\mathrm{SL}_2(\mathcal{O}_{-m})}$.
Output: The dimension of the space of level J cusp forms for $\mathrm{SL}_2(\mathcal{O}_{-m})$ of weight k.

Use Algorithm 1 to compute the matrices $M(E_1), \ldots, M(E_\ell)$ and $M(g_1), \ldots, M(g_n)$, where $M(B) = \mathrm{Symm}_k(B) \otimes \mathrm{Symm}_k(\overline{B})$.
Use Algorithm 2 to compute the matrices $P(E_1), \ldots, P(E_\ell)$ and $P(g_1), \ldots, P(g_n)$, where $P(B)$ is the matrix specifying the action of B on $\mathbb{P}^1(\mathcal{O}_{-m}/_J)$.
Compute $T(E_1), \ldots, T(E_\ell)$ and $T(g_1^{-1}), \ldots, T(g_n^{-1})$, where $T(B) = P(B) \otimes M(B)$.
`Now we compute the subspaces that are invariant under the action of the edge stabiliser groups:`
Compute $\kappa(T(E_1)), \ldots, \kappa(T(E_\ell))$, where $\kappa(B) = \mathrm{kernel}(\mathrm{Id} - B)$.
Assemble the blocks $T(g_1), \ldots, T(g_n)$ into the differential matrices $d_1, \ldots, d_\ell$.
Compute the dimension b of the spanned vector space

$$\langle \kappa(T(E_1)) \cdot d_1, \ldots, \kappa(T(E_\ell)) \cdot d_\ell \rangle$$

`which is the dimension of the image of the differential` $d_1^{1,0}$.
Compute the dimension of $E_1^{2,0}$ as $t := \mathrm{cardinality}(\mathbb{P}^1(\mathcal{O}_{-m}/_J)) \cdot N_D \cdot (k+1)^2$.
Return the difference $t - b$.