

RESEARCH STATEMENT OF ALEXANDER D. RAHM

1. RESEARCH AXIS *Torsion in the homology of discrete groups*

For my projects studying torsion in the homology of discrete groups, I have several collaborators,

- Ethan Berkove (Lafayette College, Pennsylvania),
- Tuan Anh Bui (Vietnam National University / Université de Strasbourg),
- Grant Lakeland (Eastern Illinois University),
- Matthias Wendt (Universität Wuppertal),

who have joined me because of the novel technique of *torsion subcomplex reduction*, which I have developed. It is a technique for the study of discrete groups, which I have first put to work in [10] for a specific class of arithmetic groups: the Bianchi groups, for which my method has yielded all of the homology above the virtual cohomological dimension. Some elements of this technique had already been used by Soulé for a modular group [18]; and were used by Mislin and Henn as a set of ad hoc tricks. After rediscovering these ad hoc tricks, I had success in putting them into a general framework [12]. The advantage of using a systematic technique rather than a set of ad-hoc tricks, is that instead of merely helping isolated example calculations, it becomes possible to find general formulae, as I did for instance for all of the Bianchi groups for all of the homology above their virtual cohomological dimension.

It is convenient to give some examples of where the technique of torsion subcomplex reduction has already produced good results:

- The Bianchi groups,
- The Coxeter groups,
- The PSL_2 groups over arbitrary number rings.

The Bianchi groups. In the case of the Bianchi groups (the PSL_2 groups over rings of imaginary quadratic integers), the torsion subcomplex reduction technique has permitted me to find a description of the cohomology ring of these groups in terms of elementary number-theoretic quantities [12]. The decisive step has been to extract, using torsion subcomplex reduction, the essential information about the geometric models, and then to detach this information completely from the model. I was hence able to show that this information is contained in *conjugacy classes graphs*, which I construct for this purpose for an arbitrary group from its system of conjugacy classes of finite subgroups.

The Coxeter groups. Recall that the Coxeter groups are generated by reflections; and their homology consists uniquely of torsion. Torsion subcomplex reduction hence allows to obtain all of the homology of all of the tetrahedral Coxeter groups at all odd prime numbers, both in a general formula and in terms of explicit tables [12].

The PSL_2 groups over arbitrary number rings. In joint work, Matthias Wendt and I have established formulae for the Farrell-Tate cohomology with odd torsion coefficients for all groups $\mathrm{PSL}_2(A)$, where A is a ring of S-integers in an arbitrary number field [16]. Wendt has furthermore extended this to the cases where A is the ring of functions on a smooth affine curve over an algebraically closed field. These two results together have allowed Wendt to find a refined version of the Quillen conjecture, which keeps track of all the types of known counter-examples to the original Quillen conjecture [17]. So if there does not exist any counter-example of completely new type to the original Quillen conjecture, then the Quillen–Wendt conjecture must be true.

Adaptation of the technique for SL_2 . With Ethan Berkove, I have extended my technique of torsion subcomplex reduction, which originally had been designed for groups with trivial centre (e.g., PSL_2), to be able to treat now also groups with non-trivial centre (e.g., SL_2). This way, we have determined the 2-torsion in the cohomology of the SL_2 groups over the imaginary quadratic number rings [3].

Congruence subgroups. With Ethan Berkove and Grant Lakeland (Eastern Illinois University), I have provided new tools for the calculation of the torsion in the cohomology of congruence subgroups in the Bianchi groups: An algorithm for finding particularly useful fundamental domains, and an analysis of the equivariant spectral sequence combined with torsion subcomplex reduction [2].

Extension of the technique to higher rank arithmetic groups. For machine calculations of Farrell–Tate or Bredon (co)homology, one needs cell complexes where cell stabilizers fix their cells pointwise. Tuan Anh Bui, Matthias Wendt and I have provided two algorithms computing an efficient subdivision of a complex to achieve this rigidity property [5]. Applying these algorithms to available cell complexes for $PSL_4(\mathbb{Z})$, we have computed the Farrell–Tate cohomology for small primes as well as the Bredon homology for the classifying spaces of proper actions with coefficients in the complex representation ring. In work in progress, we have established formulas for the Farrell–Tate cohomology at odd primes for GL_3 over the rings of imaginary quadratic integers.

Verification of the Quillen conjecture in the rank 2 imaginary quadratic case. With Tuan Anh Bui, I did confirm a conjecture of Quillen in the case of the mod 2 cohomology of arithmetic groups $SL_2(\mathcal{O}_{\mathbb{Q}(\sqrt{-m})}[\frac{1}{2}])$, where $\mathcal{O}_{\mathbb{Q}(\sqrt{-m})}$ is an imaginary quadratic ring of integers. To make explicit the free module structure on the cohomology ring conjectured by Quillen, we computed the mod 2 cohomology of $SL_2(\mathbb{Z}[\sqrt{-2}][\frac{1}{2}])$ via the amalgamated decomposition of the latter group [4].

Application to equivariant K -homology. In the paper [13], I have, for the Bianchi groups, transplanted the torsion subcomplex reduction technique from group homology to Bredon homology with coefficients in the complex representation rings, and with respect to the family of finite subgroups. This has lead me to formulae for this Bredon homology, and by the Atiyah–Hirzebruch spectral sequence, to formulae for equivariant K -homology of the Bianchi groups acting on their classifying space for proper actions. As the Baum–Connes assembly map from the equivariant K -homology to the K -theory of the reduced C^* -algebras of the Bianchi groups is an isomorphism, I obtain the isomorphism type of the latter operator K -theory, which would be extremely hard to compute directly from its definition.

Investigating beyond the range of arithmetic groups, my collaborators Jean-Francois Lafont (Ohio State University), Ivonne Ortiz (Miami University, Oxford, Ohio), Ruben Sanchez-Garcia (University of Southampton) and I have established formulas for the integral Bredon homology and equivariant K -homology of all compact 3-dimensional hyperbolic reflection groups [8]. In that paper, I also have proven a novel criterion for torsion-freeness of equivariant K -homology in a more general framework.

2. CONSTRUCTION OF EXPLICIT ELEMENTS IN ALGEBRAIC K-THEORY

With Rob de Jeu (VU University Amsterdam), I am searching for explicit generators for the algebraic K -theory of rings of imaginary quadratic integers. For this purpose, I have constructed non-trivial elements in the homology groups

$$H_3(\mathrm{GL}_2(\mathbb{Q}(\sqrt{-m}))),$$

in terms of geometric images coming from the Bianchi groups, and they can be lifted to algebraic K -theory. The mathematical research institute Oberwolfach did invite us with two supplementary collaborators, Herbert Gangl (Durham University) and Dan Yasaki (University of North Carolina at Greensboro), for a *Research in Pairs* project, permitting us to work on an extension of our method towards the algebraic K -theory of rings of integers in arbitrary number fields [6].

3. BIANCHI MODULAR FORMS

For my investigations on Bianchi modular forms, I did obtain a grant of 900,000 processor hours at the ICHEC (Irish Centre for High-End Computing). Bianchi modular forms are associated to tessellations of hyperbolic 3-space by Bianchi groups, as determined by diagrams like the one shown in Figure 1. Those of the cuspidal Bianchi modular forms which are relatively well understood, namely (twists of) base-change forms and CM-forms, are what we call non-genuine forms; the remaining forms are what we call *genuine*. In my paper with Mehmet Haluk Şengün (University of Sheffield) [14], an extreme paucity of genuine forms has been reported, but those and other computations were restricted to level One. My recent paper with Panagiotis Tsaknias (Université du Luxembourg) [15] extends the formulas for the non-genuine Bianchi modular forms to higher levels, and spots the first, rare instances of genuine forms at higher level and higher weight.

4. CHEN–RUAN ORBIFOLD COHOMOLOGY

With Fabio Perroni (University of Trieste), I have proved Ruan’s cohomological crepant resolution conjecture for all complexified Bianchi orbifolds, which identifies the cohomology ring structure on a crepant resolution of a complexified Bianchi orbifold (with quantum corrections) with the Chen–Ruan cohomology of the complexified Bianchi orbifold. I also have established dimension formulas for the latter [9].

5. TOPOLOGICAL DATA ANALYSIS

Topology of networks of hard rods in soft matter physics. In collaboration with Tanja Schilling (University of Freiburg)’s research group, I have investigated the electrical conductivity of networks of hard rods as a function of the volume fraction for two tunneling conductance models. For a simple, orientationally independent tunneling model, we have observed nonmonotonic behavior of the bulk conductivity as a function of volume fraction at the isotropic-nematic transition. However, this effect is lost if one allows for anisotropic tunneling. The relative conductivity enhancement increases exponentially with the volume fraction in the nematic phase. Moreover, our paper [1] contains the observation that the orientational ordering of the rods in the nematic phase induces an anisotropy in the

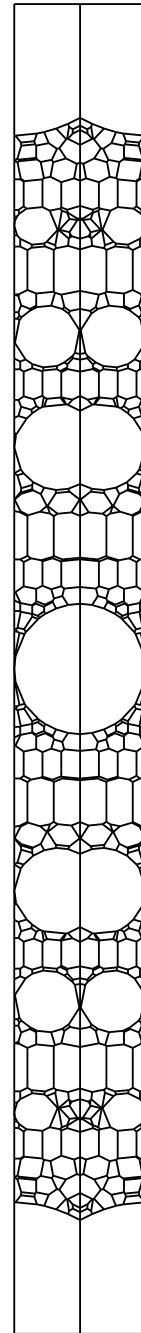


FIGURE
1. Fundamental domain for $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-102}])$, computed with [11].

conductivity, i.e., enhanced values in the direction of the nematic director field. I have used the topology of the simulated network of hard rods by computing the mesh number of the Kirchhoff network, which provides a qualitative estimate for the electrical conductivity, hence avoiding expensive computations on these large systems.

Measuring co-evolvability on protein interaction networks. With Tim Downing (Dublin City University) and further collaborators, I am searching for scale-persistent measures on the Vietoris–Rips complexes over protein interaction networks recorded in bacterial genomes, in order predict the co-evolvability between the bacterial chromosome and plasmids which provide resistance to antibiotics [7].

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- Dr. Alexander D. Rahm, Professeur à l’Université de la Polynésie Française, Laboratoire GAATI. Adjunct Lecturer at National University of Ireland. <http://gaati.org/rahm/>