# ON THE GENERALIZATIONS OF THE BRAUER–SIEGEL THEOREM

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Abstract. The classical Brauer–Siegel theorem states that if $k$ runs through the sequence of normal extensions of $\mathbb{Q}$ such that $n_k/\log|D_k| \to 0$, then $\log(h_k R_k)/\log\sqrt{|D_k|} \to 1$. In this paper we give a survey of various generalizations of this result including some recent developements in the study of the Brauer–Siegel ratio in the case of higher dimensional varieties over global fields. We also present a proof of a higher dimensional version of the Brauer–Siegel theorem dealing with the study of the asymptotic properties of the residue at $s = d$ of the zeta function in a family of varieties over finite fields.

## 1. Introduction

Let $K$ be an algebraic number field of degree $n_K = [K : \mathbb{Q}]$ and discriminant $D_K$. We define the genus of $K$ as $g_K = \log\sqrt{D_K}$. By $h_K$ we denote the class-number of $K$, $R_K$ denotes its regulator. We call a sequence $\{K_i\}$ of number fields a family if $K_i$ is non-isomorphic to $K_j$ for $i \neq j$. A family is called a tower if also $K_i \subset K_{i+1}$ for any $i$. For a family of number fields we consider the limit

$$\mathrm{BS}(\mathcal{K}) := \lim_{i\to\infty} \frac{\log(h_{K_i} R_{K_i})}{g_{K_i}}.$$

The classical Brauer–Siegel theorem, proved by Brauer (see [3]) can be stated as follows:

**Theorem 1.1** (Brauer–Siegel)**.** *For a family $\mathcal{K} = \{K_i\}$ we have*

$$\mathrm{BS}(\mathcal{K}) := \lim_{i\to\infty} \frac{\log(h_{K_i} R_{K_i})}{g_{K_i}} = 1$$

*if the family satisfies two conditions:*

(i) $\lim\limits_{i\to\infty} \frac{n_{K_i}}{g_{K_i}} = 0$;

(ii) *either the generalized Riemann hypothesis (GRH) holds, or all the fields $K_i$ are normal over $\mathbb{Q}$.*

The initial motivation for the Brauer–Siegel theorem can be traced back to a conjecture of Gauss:

**Conjecture 1.2** (Gauss)**.** *There are only 9 imaginary quadratic fields with class number equal to one, namely those having their discriminants equal to $-3, -4, -7, -8, -11, -19, -43, -67, -163$.*

The first result towards this conjecture was proven by Heilbronn in [11]. He proved that $h_K \to \infty$ as $D_K \to -\infty$. Moreover, together with Linfoot [12] he was able to verify that Gauss' list was complete with the exception of at most one discriminant. However, this "at most one" part was completely ineffective. The initial question of Gauss was settled independently by Heegner [10], Stark [28] and Baker [1] (initially the paper by Heegner was not acknowledged as giving the complete proof). We refer to [35] for a more thorough discussion of the history of the Gauss class number problem.

A natural question was to find out what happens with the class number in the case of arbitrary number fields. Here the situation is more complicated. In particular a new invariant comes into play: the regulator of number fields, which is very difficult to separate from the class number in asymtotic considerations (in particular, for this reason the other conjecture of Gauss on the infinitude of real quadratic fields having class number one is still unproven). A major step in this direction was made by Siegel [27] who was able to prove Theorem 1.1 in the case of quadratic fields. He was followed by Brauer [3] who actually proved what we call the classical Brauer–Siegel theorem.

Ever since a lot of different aspects of the problem have been studied. For example, the major difficulty in applying the Brauer–Siegel theorem to the class number problem is its ineffectiveness. Thus many attempts to obtain good explicit bounds on $h_K R_K$ were undertaken. In particular we should mention the important paper of Stark [29] giving an explicit version of the Brauer–Siegel theorem in the case when the field contains no quadratic subfields. See also some more recent papers by Louboutin [21], [22] where better explicit bounds are proven in certain cases. Even stronger effective results were needed to solve (at least in the normal case) the class-number-one problem for CM fields, see [15], [25], [2].

In another direction, assuming the generalized Riemann hypothesis (GRH) one can obtain more precise bounds on the class number then those given by the Brauer–Siegel theorem. For example in the case of quadratic fields we have $h_K << D_K^{1/2}(\log\log D_K/\log D_K)$. In particular they are known to be optimal in many cases (see [5], [6], [4]).

A full survey of the problems stemming from the study of the Brauer–Siegel type questions definitely lies beyond the scope of this article. Our goal is more modest. Here we survey the results that generalize the classical Brauer–Siegel theorem. In §2 the case of families of number fields violating one (or both) of the conditions (i) and (ii) of theorem 1.1 is discussed. In particular we introduce the notion of Tsfasman–Vlăduţ invariants of global fields that allow to express the Brauer–Siegel limit in general. In §3 we survey the known results and conjectures about the Brauer–Siegel type statements in the higher dimensional situation. Finally, in the last §4 we prove a Brauer–Siegel type result (theorem 3.2) for families of varieties over finite fields. This theorem expresses the asymptotic properties of the residue at $s = d$ of the zeta function of smooth projective varieties over finite fields via the asymptotics of the number of $\mathbb{F}_{q^m}$-points on them.

## 2. The case of global fields: Tsfasman–Vlăduţ approach

A natural question is whether one can weaken the conditions (i) and (ii) of theorem 1.1. The first condition seems to be the most restrictive one. Tsfasman and Vlăduţ were able to deal with it first in the function field case [31], [32] and then in the number field case [33] (which was as usual more difficult, especially from the analytical point of view). It turned out that one has to take in account non-archimedian place to be able to treat the general situation. Let us introduce the necessary notation in the number field case (for the function field case see §3).

For a prime power $q$ we set

$$\Phi_q(K_i) := |\{v \in P(K_i) : \mathrm{Norm}(v) = q\}|,$$

where $P(K_i)$ is the set of non-archimedian places of $K_i$. Taking in account the archimedian places we also put $\Phi_{\mathbb{R}}(K_i) = r_1(K_i)$ and $\Phi_{\mathbb{C}}(K_i) = r_2(K_i)$, where $r_1$ and $r_2$ stand for the number of real and (pairs of) complex embeddings.

We consider the set $A = \{\mathbb{R}, \mathbb{C}; 2, 3, 4, 5, 7, 8, 9, \dots\}$ of all prime powers plus two auxiliary symbols $\mathbb{R}$ and $\mathbb{C}$ as the set of indices.

**Definition 2.1.** *A family $\mathcal{K} = \{K_i\}$ is called asymptotically exact if and only if for any $\alpha \in A$ the following limit exists:*

$$\phi_\alpha = \phi_\alpha(\mathcal{K}) := \lim_{i\to\infty} \frac{\Phi_\alpha(K_i)}{g_{K_i}}.$$

*We call an asymptotically exact family $\mathcal{K}$ asymptotically good (respectively, bad) if there exists $\alpha \in A$ with $\phi_\alpha > 0$ (respectively, $\phi_\alpha = 0$ for any $\alpha \in A$). The $\phi_\alpha$ are called the Tsfasman–Vlăduţ invariants of the family $\{K_i\}$.*

One knows that any family of number fields contains an asymptotically exact subfamily so the condition on a family to be asymptotically exact is not very restrictive. On the other hand, the condition of asymptotical goodness is indeed quite restrictive. It is easy to see that a family is asymptotically bad if and only if it satisfies the condition (i) of the classical Brauer–Siegel theorem. In fact, before the work of Golod and Shafarevich [9] even the existence of asymptotically good families of number fields was unclear. Up to now the only method to construct asymptotically good families in the number field case is essentially based on the ideas of Golod and Shafarevich and consists of the usage of classfield towers (quite often in a rather elaborate way). This method has the disadvantage of beeing very inexplicit and the resulting families are hard to controll (ex. splitting of the ideals, ramification, etc.). In the function field case we dispose of a much wider range of constructions such as the towers coming from supersingular points on modular curves or Drinfeld modular curves ([16], [34]), the explicit iterated towers proposed by Garcia and Stichtenoth [7], [8] and of course the classfield towers as in the number field case (see [26] for the treatement of the function field case).

This partly explains why so little is known about the above set of invariants $\phi_\alpha$. Very few general results about the structure of the set of possible values of $(\phi_\alpha)$ are available. For instance, we do not know whether the set $\{\alpha \mid \phi_\alpha \neq 0\}$ can be infinite for some family $\mathcal{K}$. We refer to [20] for an exposition of most of the known results on the invariants $\phi_\alpha$.

Before formulating the generalization of the Brauer–Siegel theorem proven by Tsfasman and Vlăduţ in [33] we have to give one more definition. We call a number field almost normal if there exists a finite tower of number fields $\mathbb{Q} = K_0 \subset K_1 \subset \cdots \subset K_m = K$ such that all the extensions $K_i/K_{i-1}$ are normal.

**Theorem 2.2** (Tsfasman–Vlăduţ)**.** *Assume that for an asymptotically good tower $\mathcal{K}$ any of the following conditions is satisfied:*

- *GRH holds*
- *All the fields $K_i$ are almost normal over $\mathbb{Q}$.*

*Then the limit* $\mathrm{BS}(\mathcal{K}) = \lim\limits_{i\to\infty} \frac{\log(h_{K_i} R_{K_i})}{g_{K_i}}$ *exists and we have:*

$$\mathrm{BS}(\mathcal{K}) = 1 + \sum_q \phi_q \log \frac{q}{q-1} - \phi_{\mathbb{R}} \log 2 - \phi_{\mathbb{C}} \log 2\pi,$$

*the sum beeing taken over all prime powers $q$.*

We see that in the above theorem both the conditions (i) and (ii) of the classical Brauer–Siegel theorem are weakend. A natural supplement to the above theorem is the following result obtained by the author in [36]:

**Theorem 2.3** (Zykin)**.** *Let $\mathcal{K} = \{K_i\}$ be an asymptotically bad family of almost normal number fields (i. e. a family for which $n_{K_i}/g_{K_i} \to 0$ as $i \to \infty$). Then we have* $\mathrm{BS}(\mathcal{K}) = 1$.

One may ask if the values of the Brauer–Siegel ratio $\mathrm{BS}(\mathcal{K})$ can really be different from one. The answer is "yes". However, due to our lack of understanding of the set of possible $(\phi_\alpha)$ there are only partial results. Under GRH one can prove (see [33]) the following bounds on $\mathrm{BS}(\mathcal{K})$ : $0.5165 \leq \mathrm{BS}(\mathcal{K}) \leq 1.0938$. The existence bounds are weaker. There is an example of a (class field) tower with $0.5649 \leq \mathrm{BS}(\mathcal{K}) \leq 0.5975$ and another one with $1.0602 \leq \mathrm{BS}(\mathcal{K}) \leq 1.0938$ (see [33] and [36]). Our inability to get the exact value of $\mathrm{BS}(\mathcal{K})$ lies in the inexplicitness of the construction: as it was said before, class field towers are hard to control. A natural question is whether all the values of $\mathrm{BS}(\mathcal{K})$ between the bounds in the examples are attained. This seems difficult to prove at the moment though one may hope that some density results (i. e. the density of the values of $\mathrm{BS}(\mathcal{K})$ in a certain interval) are within reach of the current techniques.

Let us formulate yet another version of the generalized Brauer–Siegel theorem proven by Lebacque in [19]. It assumes GRH but has the advantage of beeing explicit in a certain (unfortunately rather weak) sense:

**Theorem 2.4** (Lebacque)**.** *Let $\mathcal{K} = \{K_i\}$ be an asymptotically exact family of number fields. Assume that* GRH *in true. Then the limit* $\mathrm{BS}(\mathcal{K})$ *exists, and we have:*

$$\sum_{q \leq x} \phi_q \log \frac{q}{q-1} - \phi_{\mathbb{R}} \log 2 - \phi_{\mathbb{C}} \log 2\pi = \mathrm{BS}(\mathcal{K}) + \mathcal{O}\left(\frac{\log x}{\sqrt{x}}\right).$$

This theorem is an easy corollary of the generalised Mertens theorem proven in [19]. We should also note that Lebacque's apporoach leads to a unified proof of theorems 2.2 and 2.3 with or without the assumption of GRH.

## 3. Varieties over global fields

Once we are in the realm of higher dimensional varieties over global fields the question of finding a proper analogue of the Brauer–Siegel theorem becomes more complicated and the answers which are currently available are far from being complete. Here we have essentially three approaches: the one by the author (which leads to a fairly simple result), another one by Kunyavskii and Tsfasman and the last one by Hindry and Pacheko (which for the moment gives only plausible conjectures). We will present all of them one by one.

The proof of the cassical Brauer–Siegel theorem as well as those of its generalisations discussed in the previous section passes through the residue formula. Let $\zeta_K(s)$ be the Dedekind zeta function of a number field $K$ and $\varkappa_K$ its residue at $s = 1$. By $w_K$ we denote the number of roots of unity in $K$. Then we have the following classical residue formula:

$$\varkappa_K = \frac{2^{r_1}(2\pi)^{r_2} h_K R_K}{w_K \sqrt{D_K}}.$$

This formula immediately reduces the proof of the Brauer–Siegel theorem to an appropriate asymptotical estimate for $\varkappa_K$ as $K$ varies in a family (by the way, this makes clear the connection with GRH which appears in the statement of the Brauer–Siegel theorem). So, in the higher dimensional situation we face two completely different problems:

(i) Study the asymptotic properties of a value of a certain $\zeta$ or $L$-function.
(ii) Find an (arithmetic or geometric) interpretation of this value.

One knows that just like in the case of global fields in the $d$-dimensional situation zeta function $\zeta_X(s)$ of a variety $X$ has a pole of order one at $s = d$. Thus the first idea would be to take the residue of $\zeta_X(s)$ at $s = d$ and study its asymptotic behaviour. In this direction we can indeed obtain a result. Let us proceed more formally.

Let $X$ be a complete non-singular absolutely irreducible projective variety of dimension $d$ defined over a finite field $\mathbb{F}_q$ with $q$ elements, where $q$ is a power of $p$. Denote by $|X|$ the set of closed points of $X$. We put $X_n = X \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}$ and $\overline{X} = X \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q$. Let $\Phi_{q^m}$ be the number of places of $X$ having degree $m$, that is $\Phi_{q^m} = |\{\mathfrak{p} \in |X| \mid \deg(\mathfrak{p}) = m\}|$. Thus the number $N_n$ of $\mathbb{F}_{q^n}$-points of the variety $X_n$ is equal to

$$N_n = \sum_{m|n} m\Phi_{q^m}.$$

Let $b_s(X) = \dim_{\mathbb{Q}_l} H^s(\overline{X}, \mathbb{Q}_l)$ be the $l$-adic Betti numbers of $X$. We set $b(X) = \max_{i=1\ldots 2d} b_i(X)$. Recall that the zeta function of $X$ is defined for $\mathrm{Re}(s) > d$ by the following Euler product:

$$\zeta_X(s) = \prod_{\mathfrak{p}\in|X|} \frac{1}{1 - N(\mathfrak{p})^{-s}} = \prod_{m=1}^{\infty} \left(\frac{1}{1 - q^{-sm}}\right)^{\Phi_{q^m}},$$

where $N(\mathfrak{p}) = q^{-\deg \mathfrak{p}}$. It is known that $\zeta_X(s)$ has an analytic continutation to a meromorphic function on the complex plane with a pole of order one at

$s = d$. Furthermore, if we set $Z(X, q^{-s}) = \zeta_X(s)$ then the function $Z(X, t)$ is a rational function of $t = q^{-s}$.

Consider a family $\{X_j\}$ of complete non-singular absolutely irreducible $d$–dimensional projective varieties over $\mathbb{F}_q$. We assume that the families under consideration satisfy $b(X_j) \to \infty$ when $j \to \infty$. Recall (see [18]) that such a family is called asymptotically exact if the following limits exist:

$$\phi_{q^m}(\{X_j\}) = \lim_{j\to\infty} \frac{\Phi_{q^m}(X_j)}{b(X_j)}, \qquad m = 1, 2, \ldots$$

The invariants $\phi_{q^m}$ of a family $\{X_j\}$ are called the Tsfasman–Vlăduţ invariants of this family. One knows that any family of varieties contains an asymptotically exact subfamily.

**Definition 3.1.** *We define the Brauer–Siegel ratio for an asymptotically exact family as*

$$\mathrm{BS}(\{X_j\}) = \lim_{j\to\infty} \frac{\log |\varkappa(X_j)|}{b(X_j)},$$

*where $\varkappa(X_j)$ is the residue of $Z(X_j, t)$ at $t = q^{-d}$.*

In §4 we prove the following generalization of the classical Brauer–Siegel theorem:

**Theorem 3.2.** *For an asymptotically exact family $\{X_j\}$ the limit $\mathrm{BS}(\{X_j\})$ exists and the following formula holds:*

$$\mathrm{BS}(\{X_j\}) = \sum_{m=1}^{\infty} \phi_{q^m} \log \frac{q^{md}}{q^{md} - 1}. \tag{1}$$

However, we come across a problem when we trying to carry out the second part of the strategy sketched above. There seems to be no easy geometric interpretaion of the invariant $\varkappa(X)$ (apart from the case $d = 1$ where we have a formula relating $\varkappa_X$ to the number of $\mathbb{F}_q$-points on the Jacobian of $X$). See however [23] for a certain cohomological interpretation of $\varkappa(X)$.

Let us now switch our attention to the two other approaches by Kunyavskii–Tsfasman and by Hindry–Pacheko. Both of them have for their starting points the famous Birch–Swinnerton-Dyer conjecture which expresses the value at $s = 1$ of the $L$-function of an abelian variety in terms of certain arithmetic invariants related to this variety. Thus, in this case we have (at least conjecturally) an interpretation of the special value of the $L$-function at $s = 1$. However, the situation with the asymptotic behaviour of this value is much less clear. Let us begin with the approach of Kunyavskii–Tsfasman. To simplify our notation we restrict ourselves to the case of elliptic curves and refer for the general case of abelian varieties to the original paper [17].

Let $K$ be a global field that is either a number field or $K = \mathbb{F}_q(X)$ where $X$ is a smooth, projective, geometrically irreducible curve over a finite field $\mathbb{F}_q$. Let $E/K$ be an elliptic curve over $K$. Let $Ш := |Ш(E)|$ be the order of the Shafarevich–Tate group of $E$, and $\Delta$ the determinant of the Mordell–Weil lattice of $E$ (see [30] for definitions). Note that in a certain sense Ш and $\Delta$ are the analogues of the class number and of the regulator respectively. The goal of Kunyavskii and Tsfasman in [17] is to study the asymptotic behaviour of the product $Ш \cdot \Delta$ as $g \to \infty$. They are able to treat the so-called constant case:

**Theorem 3.3** (Kunyavskii–Tsfasman). *Let $E = E_0 \times_{\mathbb{F}_q} K$ where $E_0$ a fixed elliptic curve over $\mathbb{F}_q$. Let $K$ vary in an asymptotically exact family $\{K_i\} = \{\mathbb{F}_q(X_i)\}$, and let $\phi_{q^m} = \phi_{q^m}(\{X_i\})$ be the corresponding Tsfasman–Vlăduţ invariants. Then*

$$\lim_{i\to\infty} \frac{\log_q(Ш_i \cdot \Delta_i)}{g_i} = 1 - \sum_{m=1}^{\infty} \phi_{q^m} \log_q \frac{N_m(E_0)}{q^m},$$

*where $N_m(E_0) = |E_0(\mathbb{F}_{q^m})|$.*

Note that there is no need to assume the above mentioned Birch and Swinnerton-Dyer conjecture as it was proven by Milne [24] in the constant case. The proof of the above theorem uses this result of Milne to get an explicit formula for $Ш \cdot \Delta$ thus reducing the proof of the theorem to the study of asymptotic properties of curves over finite fields the latter ones being much better known.

Kunyavskii and Tsfasman also make a conjecture in a certain non constant case. To formulate it we have to introduce some more notation. Let $E$ be again an arbitrary elliptic $K$-curve. Denote by $\mathcal{E}$ the corresponding elliptic surface (this means that there is a proper connected smooth morphism $f\colon \mathcal{E} \to X$ with the generic fibre $E$). Assume that $f$ fits into an infinite Galois tower, i.e. into a commutative diagram of the following form:

$$\begin{array}{ccccccccc} \mathcal{E} = \mathcal{E}_0 & \longleftarrow & \mathcal{E}_1 & \longleftarrow & \dots & \longleftarrow & \mathcal{E}_j & \longleftarrow & \dots \\ \downarrow{\scriptstyle f} & & \downarrow & & & & \downarrow & & \\ X = X_0 & \longleftarrow & X_1 & \longleftarrow & \dots & \longleftarrow & X_j & \longleftarrow & \dots, \end{array} \tag{2}$$

where each lower horizontal arrow is a Galois covering. For every $v \in X$ closed point in $X$, let $E_v = f^{-1}(v)$. Let $\Phi_{v,i}$ denote the number of points of $X_i$ lying above $v$, $\phi_v = \lim_{i\to\infty} \Phi_{v,i}/g_i$ (we suppose the limits exist). Furthermore, denote by $f_{v,i}$ the residue degree of a point of $X_i$ lying above $v$ (the tower being Galois, this does not depend on the point), and let $f_v = \lim_{i\to\infty} f_{v,i}$. If $f_v = \infty$, we have $\phi_v = 0$. If $f_v$ is finite, denote by $N(E_v, f_v)$ the number of $\mathbb{F}_{q^{f_v}}$-points of $E_v$. Finally, let $\tau$ denote the "fudge" factor in the Birch and Swinnerton-Dyer conjecture (see [30] for its precise definition). Under this setting Kunyavskii and Tsfasman formulate the following conjecture in [17]:

**Conjecture 3.4** (Kunyavskii–Tsfasman). *Assuming the Birch and Swinnerton-Dyer conjecture for elliptic curves over function fields, we have*

$$\lim_{i\to\infty} \frac{\log_q(Ш_i \cdot \Delta_i \cdot \tau_i)}{g_i} = 1 - \sum_{v\in X} \phi_v \log_q \frac{N(E_v, f_v)}{q^{f_v}}.$$

Let us finally turn our attention to the approach of Hindry and Pacheko. They treat the case in some sense "orthogonal" to that of Kunyavskii and Tsfasman. Here, contrary to the previous setting of this section, we consider the number field case as the more complete one. We refer to [14] for the function field case. As in the approach of Kunyavskii and Tsfasman we study elliptic curves over global fields. However, here the ground field $K$ is fixed and we let vary the elliptic curve $E$. Denote by $h(E)$ the logarithmic height of an elliptic curve $E$ (see [13] for the precise definition, asymptotically its properties are close to those of the conductor). Hindry in [13] formulates the following conjecture:

**Conjecture 3.5** (Hindry–Pacheko)**.** *Let $E_i$ run through a family of pairwise non-isomorphic elliptic curves over a fixed number field $K$. Then*

$$\lim_{i\to\infty} \frac{\log(\text{Ш}_i \cdot \Delta_i)}{h(E_i)} = 1.$$

To motivate this conjecture, Hidry reduces it to a conjecture on the asymptotics of the special value of $L$-functions of elliptic curves at $s = 1$ using the conjecture of Birch and Swinnerton-Dyer as well as that of Szpiro and Frey (the latter one is equivalent to the ABC conjecture when $K = \mathbb{Q}$).

Let us finally state some open questions that arise naturally from the above discussion.

- What is the number field analogue of theorem 3.2?

It seems not so difficult to prove the result corresponding to theorem 3.2 in the number field case assuming GRH. Without GRH the situation looks much more challenging. In particular, one has to be able to controll the so called Siegel zeroes of zeta functions of varieties (that is real zeroes close to $s = d$) which might turn out to be a difficult problem. The conjecture 3.4 can be easily written in the number field case. However, in this situation we have even less evidence for it since theorem 3.3 is a particular feature of the function field case.

- How can one unify the conjectures of Kunyavskii–Tsfasman and Hindry—Pacheko?

In particular it is unclear which invariant of elliptic curves should play the role of genus from the case of global fields. It would also be nice to be able to formulate some conjectures for a more general type of $L$-functions, such as automorphic $L$-functions.

- Is it possible to justify any of the above conjectures in certain particular cases? Can one prove some cases of these conjectures "on average" (in some appropriate sense)?

For now the only case at hand is the one given by theorem 3.3.

## 4. The proof of the Brauer–Siegel theorem for varieties over finite fields: case $s = d$

Recall that the trace formula of Lefschetz–Grothendieck gives the following expression for $N_n$ — the number of $\mathbb{F}_{q^n}$ points on a variety $X$ :

$$N_n = \sum_{s=0}^{2d} (-1)^s q^{ns/2} \sum_{i=1}^{b_s} \alpha_{s,i}^n, \tag{3}$$

where $\{q^{s/2}\alpha_{s,i}\}$ is the set of of inverse eigenvalues of the Frobenius endomorphism acting on $H^s(\overline{X}, \mathbb{Q}_l)$. By Poincaré duality one has $b_{2d-s} = b_s$ and $\alpha_{s,i} = \alpha_{2d-s,i}$. The conjecture of Riemann–Weil proven by Deligne states that the absolute values of $\alpha_{s,i}$ are equal to 1. One also knows that $b_0 = 1$ and $\alpha_{0,1} = 1$.

One can easily see that for $Z(X, q^{-s}) = \zeta_X(s)$ we have the following power series expansion:

$$\log Z(X,t) = \sum_{n=1}^{\infty} N_n \frac{t^n}{n}. \tag{4}$$

Combining (4) and (3) we obtain

$$Z(X,t) = \prod_{s=0}^{2d} (-1)^{s-1} P_s(X,t), \tag{5}$$

where $P_s(X,t) = \prod_{i=1}^{b_i}(1 - q^{s/2}\alpha_{s,i})$. Furthermore we note that $P_0(X,t) = 1 - t$ and $P_{2d}(X,t) = 1 - q^d t$.

To prove theorem 3.2 we will need the following lemma.

**Lemma 4.1.** *For $c \to \infty$ we have*

$$\frac{\log|\varkappa(X_j)|}{b(X_j)} = \sum_{l=1}^{c} \frac{N_l(X_j) - q^{dl}}{l} q^{-dl} + R_c(X_j),$$

*with $R_c(X_j) \to 0$ uniformly in $j$.*

*Proof of the Lemma.* Using (5) one has

$$\frac{\log|\varkappa(X_j)|}{b(X_j)} + d\frac{\log q}{b(X_j)} = \frac{1}{b(X_j)} \sum_{s=0}^{2d-1} (-1)^{s+1} \log|P_s(X_j, q^{-d})| =$$
$$= \frac{1}{b(X_j)} \sum_{s=0}^{2d-1} (-1)^{s+1} \sum_{k=1}^{b_s(X_j)} \log(1 - q^{(s-2d)/2}\alpha_{s,i}) =$$
$$= -\frac{1}{b(X_j)} \sum_{s=0}^{2d-1} (-1)^{s+1} \sum_{k=1}^{b_s(X_j)} \sum_{l=1}^{\infty} \frac{q^{(s-2d)l/2}\alpha_{s,i}^l}{l} =$$
$$= \frac{1}{b(X_j)} \sum_{l=1}^{c} \frac{q^{-dl}}{l} \left( \sum_{s=0}^{2d} (-1)^s q^{sl/2} \sum_{k=1}^{b_s(X_j)} \alpha_{s,i}^l - q^{dl} \right) +$$
$$+ \frac{1}{b(X_j)} \sum_{s=0}^{2d-1} (-1)^s \sum_{k=1}^{b_s(X_j)} \sum_{l=c+1}^{\infty} \frac{q^{(s-2d)l/2}\alpha_{s,i}^l}{l} =$$
$$= \sum_{l=1}^{c} \frac{N_l(X_j) - q^{dl}}{l} q^{-dl} + R_c(X_j).$$

An obvious estimate gives

$$|R_c(X_j)| \le \frac{\sum_{s=0}^{2d} b_s(X_j)}{b(X_j)} \sum_{l=c+1}^{\infty} \frac{q^{-l/2}}{l} \to 0$$

for $c \to \infty$ uniformly in $j$. $\square$

Now let us note that

$$\frac{1}{b(X_j)}\sum_{l=1}^{c}\frac{1}{l}\leq\frac{2}{b(X_j)}\log c\to 0$$

when $\log c/b(X_j)\to 0$. Thus to prove the main theorem we are left to deal with the following sum:

$$\frac{1}{b(X_j)}\sum_{l=1}^{c}\frac{q^{-ld}}{l}N_l(X_j)=$$

$$=\frac{1}{b(X_j)}\sum_{l=1}^{c}\frac{q^{-dl}}{l}\sum_{m|l}m\Phi_{q^m}=\frac{1}{b(X_j)}\sum_{m=1}^{c}\Phi_{q^m}\sum_{k=1}^{\lfloor c/m\rfloor}\frac{q^{-mkd}}{k}=$$

$$=\frac{1}{b(X_j)}\sum_{m=1}^{c}\Phi_{q^m}\log\frac{q^{md}}{q^{md}-1}-\frac{1}{b(X_j)}\sum_{m=1}^{c}\Phi_{q^m}\sum_{\lfloor c/m\rfloor+1}^{\infty}\frac{q^{-mkd}}{k}.$$

Let us estimate the last term:

$$\frac{1}{b(X_j)}\sum_{m=1}^{c}\Phi_{q^m}\sum_{k=\lfloor c/m\rfloor+1}^{\infty}\frac{q^{-mkd}}{k}\leq$$

$$\leq\frac{1}{b(X_j)}\sum_{m=1}^{c}\frac{N_m(X_j)q^{-md(\lfloor c/m\rfloor+1)}}{m(\lfloor c/m\rfloor+1)(1-q^{-md})}\leq\frac{1}{b(X_j)}\sum_{m=1}^{c}\frac{N_m(X_j)q^{-cd}}{c(1-q^{-md})}\leq$$

$$\leq\frac{1}{b(X_j)}\sum_{m=1}^{c}\left(q^{md}+1+\sum_{s=1}^{2d-1}b_sq^{ms/2}\right)\frac{q^{-dc}}{c(1-q^{-md})}\leq$$

$$\leq\frac{1}{b(X_j)}\left(q^{cd}+1+\sum_{s=1}^{2d-1}b_sq^{cs/2}\right)\frac{q^{-dc}}{(1-q^{-1})}\to 0$$

as both $b(X_j)\to\infty$ and $c\to\infty$.

Now, to finish the proof we will need an analogue of the basic inequality from [31]. In the higher dimensional case there are several versions of it. However, here the simplest one will suffice. Let us define for $i=0\ldots 2d$ the following invariants:

$$\beta_i(\{X_j\})=\limsup_{j}\frac{b_i(X_j)}{b(X_j)}.$$

**Theorem 4.2.** *For an asymptotically exact family $\{X_j\}$ we have the inequality:*

$$\sum_{m=1}^{\infty}\frac{m\phi_{q^m}}{q^{(2d-1)m/2}-1}\leq(q^{(2d-1)/2}-1)\left(\sum_{i\equiv 1\bmod 2}\frac{\beta_i}{q^{(i-1)/2}+1}+\sum_{i\equiv 0\bmod 2}\frac{\beta_i}{q^{(i-1)/2}-1}\right).$$

*Proof.* See [18], Remark 8.8. □

Applying this theorem together with the fact that

$$\log\frac{q^{md}}{q^{md}-1}=O\left(\frac{1}{q^{dm}-1}\right)=O\left(\frac{m}{q^{(2d-1)m/2}-1}\right)$$

when $m \to \infty$, we conclude that the series on the right hand side of (1) converges. Thus the difference

$$\sum_{m=1}^{\infty} \phi_{q^m} \log \frac{q^{md}}{q^{md}-1} - \frac{1}{b(X_j)} \sum_{m=1}^{c} \Phi_{q^m} \log \frac{q^{md}}{q^{md}-1} =$$
$$= \sum_{m=1}^{c} \left( \phi_{q^m} - \frac{\Phi_{q^m}}{b(X_j)} \right) \log \frac{q^{md}}{q^{md}-1} - \sum_{m=c+1}^{\infty} \phi_{q^m} \log \frac{q^{md}}{q^{md}-1} \to 0$$

when $c \to \infty, j \to \infty$ and $j$ is large enough compared to $c$. This concludes the proof of theorem 3.2.

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